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# Bodily tides near spin-orbit resonances

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## Abstract

Spin-orbit coupling can be described in two approaches. The first method, known as the “*MacDonald torque*”, is often combined with a convenient assumption that the quality factor  $Q$  is frequency-independent. This makes the method inconsistent, for the MacDonald theory tacitly fixes the rheology of the mantle by making  $Q$  scale as the inverse tidal frequency.

Spin-orbit coupling can be treated also in an approach called “*the Darwin torque*”. While this theory is general enough to accommodate an arbitrary frequency-dependence of  $Q$ , this advantage has not yet been fully exploited in the literature, where  $Q$  is often assumed constant or is set to scale as inverse tidal frequency, the latter assertion making the Darwin torque equivalent to a corrected version of the MacDonald torque.

However neither a constant nor an inverse-frequency  $Q$  reflect the properties of realistic mantles and crusts, because the actual frequency-dependence is more complex. Hence it is necessary to enrich the theory of spin-orbit interaction with the right frequency-dependence.

We accomplish this programme for the Darwin-torque-based model near resonances. We derive the frequency-dependence of the tidal torque from the first principles of solid-state mechanics, i.e., from the expression for the mantle’s compliance in the time domain. We also explain that the tidal torque includes not only the customary, secular part, but also an oscillating part.

We demonstrate that the  $lmpq$  term of the Darwin-Kaula expansion for the tidal torque smoothly passes zero, when the secondary traverses the  $lmpq$  resonance (e.g., the principal tidal torque smoothly goes through nil as the secondary crosses the synchronous orbit).

Thus we prepare a foundation for modeling entrapment of a despinning primary into a resonance with its secondary. The roles of the primary and secondary may be played, e.g., by Mercury and the Sun, correspondingly, or by an icy moon and a Jovian planet.

We also offer a possible explanation for the unexpected frequency-dependence of the tidal dissipation rate in the Moon, discovered by LLR.

## 1 Introduction

We continue a critical examination of the tidal-torque techniques, begun in Efroimsky & Williams (2009), where the empirical treatment by and MacDonald (1964) was considered from the viewpoint of a more general and rigorous approach by Darwin (1879, 1880) and Kaula (1964). Referring the Reader to Efroimsky & Williams (2009) for proofs and comments, we begin with an inventory of the key formulae describing the spin-orbit interaction. While in *Ibid.* we employed those formulae to explore tidal despinning well outside the 1:1 resonance (and in neglect of the intermediate resonances), in the current paper we apply this machinery to the case of despinning in the vicinity of a spin-orbit resonance.

Although the topic has been popular since mid-sixties and has already been addressed in books, the common models are not entirely adequate to the actual physics. Just as in the nonresonant case discussed in *Ibid.*, a generic problem with the popular models of libration or of capture into a resonance is that they employ wrong rheologies (the work by Rambaux et al. 2010 being the only exception we know of). Above that, the model based on the MacDonald torque suffers a defect stemming from a genuine inconsistency inherent in the theory by MacDonald (1964).

As explained in Efroimsky and Williams (2009) and Williams and Efroimsky (2012), the MacDonald theory, both in its original and corrected versions, tacitly fixes an unphysical shape of the functional dependence  $Q(\chi)$ , where  $Q$  is the dissipation quality factor and  $\chi$  is the tidal frequency (Williams & Efroimsky 2012). So we base our approach on the developments by Darwin (1879, 1880) and Kaula (1964), combining those with a realistic law of frequency-dependence of the damping rate.

Since our main purpose is to lay the groundwork for the subsequent study of the process of falling into a resonance, the two principal results obtained in this paper are the following:

(a) Starting with the realistic rheological model (the expression for the compliance in the time domain), we derive the complex Love numbers  $\bar{k}_l$  as functions of the frequency  $\chi$ , and write down their negative imaginary parts as functions of the frequency:  $-\mathcal{I}m[\bar{k}_l(\chi)] = |k_l(\chi)| \sin \epsilon_l(\chi)$ . It is these expressions that appear as factors in the terms of the Darwin-Kaula expansion of tides. These factors' frequency-dependencies demonstrate a nontrivial shape, especially near resonances. This shape plays a crucial role in modeling of despinning in general, specifically in modeling the process of falling into a spin-orbit resonance.

(b) We demonstrate that, beside the customary secular part, the Darwin torque contains a usually omitted oscillating part.

## 2 Linear bodily tides

Linearity of tide means that: (a) under a static load, deformation scales linearly, and (b) under undulatory loading, the same linear law applies, separately, to each frequency mode. The latter implies that the deformation magnitude at a certain frequency should depend linearly upon the tidal stress at this frequency, and should bear no dependence upon loading at other tidal modes. Thence the dissipation rate at that frequency will depend on the stress at that frequency only.

### 2.1 Linearity of the tidal deformation

At a point  $\vec{R} = (R, \lambda, \phi)$ , the potential due to a tide-raising secondary of mass  $M_{sec}^*$ , located at  $\vec{r}^* = (r^*, \lambda^*, \phi^*)$  with  $r^* \geq R$ , is expandable over the Legendre polynomials  $P_l(\cos \gamma)$ :

$$\begin{aligned} W(\vec{R}, \vec{r}^*) &= \sum_{l=2}^{\infty} W_l(\vec{R}, \vec{r}^*) = - \frac{G M_{sec}^*}{r^*} \sum_{l=2}^{\infty} \left( \frac{R}{r^*} \right)^l P_l(\cos \gamma) \\ &= - \frac{G M_{sec}^*}{r^*} \sum_{l=2}^{\infty} \left( \frac{R}{r^*} \right)^l \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi) P_{lm}(\sin \phi^*) \cos m(\lambda - \lambda^*) \quad , \quad (1) \end{aligned}$$

where  $G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$  is Newton's gravity constant, and  $\gamma$  is the angular separation between the vectors  $\vec{r}^*$  and  $\vec{R}$  pointing from the primary's centre. The latitudes  $\phi, \phi^*$  are reckoned from the primary's equator, while the longitudes  $\lambda, \lambda^*$  are reckoned from a fixed meridian.

Under the assumption of linearity, the  $l^{th}$  term  $W_l(\vec{R}, \vec{r}^*)$  in the secondary's potential causes a linear deformation of the primary's shape. The subsequent adjustment of the primary's potential being linear in the said deformation, the  $l^{th}$  adjustment  $U_l$  of the primary's potential

is proportional to  $W_l$ . The theory of potential requires  $U_l(\vec{r})$  to fall off, outside the primary, as  $r^{-(l+1)}$ . Thus the overall amendment to the potential of the primary amounts to:

$$U(\vec{r}) = \sum_{l=2}^{\infty} U_l(\vec{r}) = \sum_{l=2}^{\infty} k_l \left( \frac{R}{r} \right)^{l+1} W_l(\vec{R}, \vec{r}^*) \quad , \quad (2)$$

$R$  now being the mean equatorial radius of the primary,  $\vec{R} = (R, \phi, \lambda)$  being a surface point,  $\vec{r} = (r, \phi, \lambda)$  being an exterior point located above it at a radius  $r \geq R$ . The coefficients  $k_l$ , called Love numbers, are defined by the primary's rheology.

For a homogeneous incompressible spherical primary of density  $\rho$ , surface gravity  $g$ , and rigidity  $\mu$ , the *static* Love number of degree  $l$  is given by

$$k_l = \frac{3}{2(l-1)} \frac{1}{1 + A_l} \quad , \quad \text{where} \quad A_l \equiv \frac{(2l^2 + 4l + 3)\mu}{lg\rho R} = \frac{3(2l^2 + 4l + 3)\mu}{4l\pi G\rho^2 R^2} \quad . \quad (3)$$

For  $R \ll r, r^*$ , consideration of the  $l = 2$  input in (2) turns out to be sufficient.<sup>1</sup>

These formulae apply to *static* deformations. However an actual tide is never static, except in the case of synchronous orbiting with a zero eccentricity and inclination.<sup>2</sup> Hence a realistic perturbing potential produced by the secondary carries a spectrum of modes  $\omega_{lmpq}$  (positive or negative) numbered with four integers  $lmpq$  as in formula (105) below. The perturbation causes a spectrum of stresses in the primary, at frequencies  $\chi_{lmpq} = |\omega_{lmpq}|$ . Although in a linear medium strains are generated exactly at the frequencies of the stresses, friction makes each Fourier component of the strain fall behind the corresponding component of the stress. Friction also reduces the magnitude of the shape response – hence the deviation of a dynamical Love number  $k_l(\chi)$  from its static counterpart  $k_l = k_l(0)$ . Below we shall explain that formulae (2 - 3) can be easily adjusted to the case of undulatory tidal loads in a homogeneous planet or in tidally-despinning homogeneous satellite (treated now as the primary, with its planet playing the role of the tide-raising secondary). However generalisation of formulae (2 - 3) to the case of a librating moon (treated as a primary) turns out to be highly nontrivial. As we shall see, the standard derivation by Love (1909, 1911) falls apart in the presence of the non-potential inertial force containing the time-derivative of the primary's angular velocity.

The frequency-dependence of a dynamical Love numbers takes its origins in the “inertia” of strain and, therefore, of the shape of the body. Hence the analogy to linear circuits: the  $l^{th}$  components of  $W$  and  $U$  act as a current and voltage, while the  $l^{th}$  Love number plays, up to a factor, the role of impedance. Therefore, under a sinusoidal load of frequency  $\chi$ , it is convenient to replace the actual Love number with its complex counterpart

$$\bar{k}_l(\chi) = |\bar{k}_l(\chi)| \exp[-i\epsilon_l(\chi)] \quad , \quad (4)$$

$\epsilon_l$  being the frequency-dependent phase delay of the reaction relative to the load (Munk & MacDonald 1960, Zschau 1978). The “minus” sign in (4) makes  $U$  lag behind  $W$  for a positive  $\epsilon_l$ . (So the situation resembles a circuit with a capacitor, where the current leads voltage.)

In the limit of zero frequency, i.e., for a steady deformation, the lag should vanish, and so should the entire imaginary part:

$$\mathcal{Im} [\bar{k}_l(0)] = |\bar{k}_l(0)| \sin \epsilon_l(0) = 0 \quad , \quad (5)$$

<sup>1</sup> Special is the case of Phobos, for whose orbital evolution the  $k_3$  and perhaps even the  $k_4$  terms may be relevant (Bills et al. 2005). Another class of exceptions is constituted by close binary asteroids. The topic is addressed by Taylor & Margot (2010), who took into account the Love numbers up to  $k_6$ .

<sup>2</sup> The case of a permanently deformed moon in a 1:1 spin-orbit resonance falls under this description too. Recall that in the tidal context the distorted body is taken to be the primary. So from the viewpoint of the satellite its host planet is orbiting the satellite synchronously, thus creating a static tide.

leaving the complex Love number real:

$$\bar{k}_l(0) = \mathcal{R}e [\bar{k}_l(0)] = |\bar{k}_l(0)| \cos \epsilon_l(0) , \quad (6)$$

and equal to the customary static Love number:

$$\bar{k}_l(0) = k_l . \quad (7)$$

Solution of the equation of motion combined with the constitutive (rheological) equation renders the complex  $\bar{k}_l(\chi)$ , as explained in Appendix D.1. Once  $\bar{k}_l(\chi)$  is found, its absolute value

$$k_l(\chi) \equiv |\bar{k}_l(\chi)| \quad (8)$$

and negative argument

$$\epsilon_l(\chi) = - \arctan \frac{\mathcal{I}m [\bar{k}_l(\chi)]}{\mathcal{R}e [\bar{k}_l(\chi)]} \quad (9)$$

should be inserted into the  $l^{th}$  term of the Fourier expansion for the tidal potential. Things get simplified when we study how the tide, caused on the primary by a secondary, is acting on that same secondary. In this case, the  $l^{th}$  term in the Fourier expansion contains  $|k_l(\chi)|$  and  $\epsilon_l(\chi)$  in the convenient combination  $k_l(\chi) \sin \epsilon_l(\chi)$ , which is exactly  $-\mathcal{I}m [\bar{k}_l(\chi)]$ .

Rigorously speaking, we should say not “the  $l^{th}$  term”, but “the  $l^{th}$  terms”, as each  $l$  corresponds to an infinite set of positive and negative Fourier modes  $\omega_{lmpq}$ , the physical forcing frequencies being  $\chi = \chi_{lmpq} \equiv |\omega_{lmpq}|$ . Thus, while the functional forms of both  $|k_l(\chi)|$  and  $\sin \epsilon_l(\chi)$  depend only on  $l$ , both functions take values that are different for different sets of numbers  $mpq$ . This happens because  $\chi$  assumes different values  $\chi_{lmpq}$  on these sets. Mind though that for triaxial bodies the functional forms of  $|k_l(\chi)|$  and  $\sin \epsilon_l(\chi)$  may depend also on  $m, p, q$ .

## 2.2 Damping of a linear tide

Beside the standard assumption  $U_l(\vec{r}) \propto W_l(\vec{R}, \vec{r}^*)$ , the linearity condition includes the requirement that the functions  $k_l(\chi)$  and  $\epsilon_l(\chi)$  be well defined. This implies that they depend solely upon the frequency  $\chi$ , and not upon the other frequencies involved. Nor shall the Love numbers or lags be influenced by the stress or strain magnitudes at this or other frequencies.

Then, at frequency  $\chi$ , the mean (over a period) damping rate  $\langle \dot{E}(\chi) \rangle$  depends on the value of  $\chi$  and on the loading at that frequency, and is not influenced by the other frequencies:

$$\langle \dot{E}(\chi) \rangle = - \frac{\chi E_{peak}(\chi)}{Q(\chi)} \quad (10)$$

or, equivalently:

$$\Delta E_{cycle}(\chi) = - \frac{2 \pi E_{peak}(\chi)}{Q(\chi)} , \quad (11)$$

$\Delta E_{cycle}(\chi)$  being the one-cycle energy loss, and  $Q(\chi)$  being the so-called quality factor.

If  $E_{peak}(\chi)$  in (10 - 11) is agreed to denote the peak *energy* stored at frequency  $\chi$ , the appropriate  $Q$  factor is connected to the phase lag  $\epsilon(\chi)$  through

$$Q_{energy}^{-1} = \sin |\epsilon| . \quad (12)$$

and *not* through  $Q_{energy}^{-1} = \tan |\epsilon|$  as often presumed (see Appendix B for explanation).

If  $E_{peak}(\chi)$  is defined as the peak *work*, the corresponding  $Q$  factor is related to the lag via

$$Q_{work}^{-1} = \frac{\tan |\epsilon|}{1 - \left(\frac{\pi}{2} - |\epsilon|\right) \tan |\epsilon|} , \quad (13)$$

as demonstrated in Appendix B below.<sup>3</sup> In the limit of a small  $\epsilon$ , (13) becomes

$$Q_{work}^{-1} = \sin |\epsilon| + O(\epsilon^2) = |\epsilon| + O(\epsilon^2) , \quad (14)$$

so definition (13) makes  $1/Q$  a good approximation to  $\sin \epsilon$  for small lags only.

For the lag approaching  $\pi/2$ , the quality factor defined through (12) attains its minimum,  $Q_{energy} = 1$ , while definition (13) furnishes  $Q_{work} = 0$ . The latter is not surprising, as in the said limit no work is carried out on the system.

Linearity requires the functions  $\bar{k}_l(\chi)$  and therefore also  $\epsilon_l(\chi)$  to be well-defined, i.e., to be independent from all the other frequencies but  $\chi$ . We now see, the requirement extends to  $Q(\chi)$ .

The third definition of the quality factor (offered by Golderich 1963) is  $Q_{Goldreich}^{-1} = \tan |\epsilon|$ . However this definition corresponds neither to the peak work nor to the peak energy. The existing ambiguity in definition of  $Q$  makes this factor redundant, and we mention it here only as a tribute to the tradition. As we shall see, all practical calculations contain the products of the Love numbers by the sines of the phase lags,  $k_l \sin \epsilon_l$ , where  $l$  is the degree of the appropriate spherical harmonic. A possible compromise between this mathematical fact and the historical tradition of using  $Q$  would be to define the quality factor through (12), in which case the quality factor must be equipped with the subscript  $l$ . (This would reflect the profound difference between the tidal quality factors and the seismic quality factor – see Efroimsky 2012.)

### 3 Several basic facts from continuum mechanics

This section offers a squeezed synopsis of the basic facts from the linear solid-state mechanics. A more detailed introduction, including a glossary and examples, is offered in Appendix A.

#### 3.1 Stationary linear deformation of isotropic incompressible media

Mechanical properties of a medium are furnished by the so-called constitutive equation or constitutive law, which interrelates the stress tensor  $\mathbb{S}$  with the strain tensor  $\mathbb{U}$  defined as

$$\mathbb{U} \equiv \frac{1}{2} \left[ (\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^T \right] , \quad (15)$$

where  $\mathbf{u}$  is the vector of displacement.

As we shall consider only linear deformations, our constitutive laws will be linear, and will be expressed by equations which may be algebraic, differential, integral, or integro-differential.

The elastic stress  $\overset{(e)}{\mathbb{S}}$  is related to  $\mathbb{U}$  through the simplest constitutive equation

$$\overset{(e)}{\mathbb{S}} = \mathbb{B} \mathbb{U} , \quad (16)$$

$\mathbb{B}$  being a four-dimensional matrix of real numbers called *elasticity moduli*.

A hereditary stress  $\overset{(h)}{\mathbb{S}}$  is connected to  $\mathbb{U}$  as

$$\overset{(h)}{\mathbb{S}} = \tilde{\mathbb{B}} \mathbb{U} , \quad (17)$$

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<sup>3</sup>Deriving this formula in Appendix to Efroimsky & Williams (2009), we inaccurately termed  $E_{peak}(\chi)$  as peak energy. However our calculation of  $Q$  was carried out in understanding that  $E_{peak}(\chi)$  is the peak *work*.

$\tilde{\mathbb{B}}$  being a four-dimensional integral-operator-valued matrix. Its component  $\tilde{B}_{ijkl}$  acts on an element  $u_{kl}$  of the strain not as a mere multiplier but as an integral operator, with integration going from  $t' = -\infty$  through  $t' = t$ . To furnish the value of  $\sigma_{ij} = \sum_{kl} \tilde{B}_{ijkl} u_{kl}$  at time  $t$ , the operator “consumes” as arguments all the values of  $u_{kl}(t')$  over the interval  $t' \in (-\infty, t]$ .

The viscous stress is related to the strain through a differential operator  $\mathbb{A} \frac{\partial}{\partial t}$ :

$$\mathbb{S}^{(v)} = \mathbb{A} \frac{\partial}{\partial t} \mathbb{U} \quad , \quad (18)$$

$\mathbb{A}$  being a four-dimensional matrix consisting of empirical constants called viscosities.

In an isotropic medium, each of the three matrices,  $\mathbb{B}$ ,  $\tilde{\mathbb{B}}$ , and  $\tilde{\mathbb{A}}$ , includes two terms only. The elastic stress becomes:

$$\mathbb{S}^{(e)} = \mathbb{S}_{volumetric}^{(e)} + \mathbb{S}_{deviatoric}^{(e)} = 3K \left( \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) + 2\mu \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) \quad , \quad (19)$$

with  $K$  and  $\mu$  being the *bulk elastic modulus* and the *shear elastic modulus*, correspondingly,  $\mathbb{I}$  standing for the unity matrix, and  $\text{Sp}$  denoting the trace of a matrix:  $\text{Sp} \mathbb{U} \equiv \sum_i U_{ii}$ .

The hereditary stress becomes:

$$\mathbb{S}^{(h)} = \mathbb{S}_{volumetric}^{(h)} + \mathbb{S}_{deviatoric}^{(h)} = 3\tilde{K} \left( \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) + 2\tilde{\mu} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) \quad , \quad (20)$$

where  $\tilde{K}$  and  $\tilde{\mu}$  are the *bulk-modulus operator* and the *shear-modulus operator*, accordingly.

The viscous stress acquires the form:

$$\mathbb{S}^{(v)} = \mathbb{S}_{volumetric}^{(v)} + \mathbb{S}_{deviatoric}^{(v)} = 3\zeta \frac{\partial}{\partial t} \left( \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) + 2\eta \frac{\partial}{\partial t} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) \quad , \quad (21)$$

the quantities  $\zeta$  and  $\eta$  being termed as the *bulk viscosity* and the *shear viscosity*, correspondingly.

The term  $\frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U}$  is called the *volumetric* part of the strain, while  $\mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U}$  is called the *deviatoric* part. Accordingly, in expressions (161 - 163) for the stresses, the pure-trace terms are called *volumetric*, the other term being named *deviatoric*.

If an isotropic medium is also incompressible, the relative change of the volume vanishes:  $\text{Sp} \mathbb{U} = 0$ , and so does the expansion rate:  $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial t} \text{Sp} \mathbb{U} = 0$ . Then the volumetric part of the strain becomes zero, and so do the volumetric parts of the elastic, hereditary, and viscous stresses. The incompressibility assumption may be applicable both to crusty objects and to large icy moons of low porosity. At least for Iapetus, the low-porosity assumption is likely to be correct (Castillo-Rogez et al. 2011).

## 3.2 Approaches to modeling viscoelastic deformations.

### Problems with terminology

One approach to linear deformations is to assume that the elastic, hereditary and viscous deviatoric stresses simply sum up, each of them being linked to the same overall deviatoric strain:

$$\mathbb{S}^{(total)} = \mathbb{S}^{(e)} + \mathbb{S}^{(h)} + \mathbb{S}^{(v)} = \mathbb{B} \mathbb{U} + \tilde{\mathbb{B}} \mathbb{U} + \mathbb{A} \frac{\partial}{\partial t} \mathbb{U} = \left( \mathbb{B} + \tilde{\mathbb{B}} + \mathbb{A} \frac{\partial}{\partial t} \right) \mathbb{U} \quad . \quad (22)$$

An alternative option, to be used in section 5.3 below, is to start with an overall deviatoric stress, and to expand the deviatoric strain into elastic, viscous, and hereditary parts:

$$\mathbb{U} = \mathbb{U}^{(e)} + \mathbb{U}^{(h)} + \mathbb{U}^{(v)} \quad , \quad \mathbb{U}^{(e)} = \frac{1}{\mu} \mathbb{S} \quad , \quad \mathbb{U}^{(v)} = \frac{1}{\eta} \int^t \mathbb{S}(t') dt' \quad , \quad \mathbb{U}^{(h)} = \tilde{J} \mathbb{S} \quad , \quad (23)$$

$\tilde{J}$  being an integral operator with a time-dependent kernel.

An even more general option would be to assume that both the strain and stress are comprised by components of different nature – elastic, hereditary, viscous, or more complicated (plastic). Which option to choose – depends upon the medium studied. The rich variety of materials offered to us by nature leaves one no chance to develop a unified theory of deformation.

As different segments of the continuum-mechanics community use different conventions on the meaning of some terms, we offer a glossary of terms in Appendix A. Here we would only mention that in our paper the term *viscoelastic* will be applied to a model containing not only viscous and elastic terms, but also an extra term responsible for an anelastic hereditary reaction. (A more appropriate term *viscoelastohereditary* would be way too cumbersome.)

### 3.3 Evolving stresses and strains. Basic notations

In the general case, loading varies in time, so one has to deal with the stress and strain tensors as functions of time. However, treatment of viscoelasticity turns out to be simpler in the frequency domain, i.e., in the language of complex rigidity and complex compliance. To this end, the stress  $\sigma_{\gamma\nu}$  and strain  $u_{\gamma\nu}$  in a linear medium can be Fourier-expanded as

$$\sigma_{\gamma\nu}(t) = \sum_{n=0}^{\infty} \sigma_{\gamma\nu}(\chi_n) \cos[\chi_n t + \varphi_{\sigma}(\chi_n)] = \sum_{n=0}^{\infty} \mathcal{R}e \left[ \sigma_{\gamma\nu}(\chi_n) e^{i\chi_n t + i\varphi_{\sigma}(\chi_n)} \right] \quad (24a)$$

$$= \sum_{n=0}^{\infty} \mathcal{R}e \left[ \bar{\sigma}_{\gamma\nu}(\chi_n) e^{i\chi_n t} \right] \quad , \quad (24b)$$

$$u_{\gamma\nu}(t) = \sum_{n=0}^{\infty} u_{\gamma\nu}(\chi_n) \cos[\chi_n t + \varphi_u(\chi_n)] = \sum_{n=0}^{\infty} \mathcal{R}e \left[ u_{\gamma\nu}(\chi_n) e^{i\chi_n t + i\varphi_u(\chi_n)} \right] \quad (25a)$$

$$= \sum_{n=0}^{\infty} \mathcal{R}e \left[ \bar{u}_{\gamma\nu}(\chi_n) e^{i\chi_n t} \right] \quad , \quad (25b)$$

where the complex amplitudes are:

$$\bar{\sigma}_{\gamma\nu}(\chi) = \sigma_{\gamma\nu}(\chi) e^{i\varphi_{\sigma}(\chi)} \quad , \quad \bar{u}_{\gamma\nu}(\chi) = u_{\gamma\nu}(\chi) e^{i\varphi_u(\chi)} \quad , \quad (26)$$

while the initial phases  $\varphi_{\sigma}(\chi)$  and  $\varphi_u(\chi)$  are chosen in a manner that sets the real amplitudes  $\sigma_{\gamma\nu}(\chi_n)$  and  $u_{\gamma\nu}(\chi_n)$  non-negative.

We wrote the above expansions as sums over a discrete spectrum, as the spectrum generated by tides is discrete. Generally, the sums can, of course, be replaced with integrals over frequency:

$$\sigma_{\gamma\nu}(t) = \int_0^{\infty} \bar{\sigma}_{\gamma\nu}(\chi) e^{i\chi t} d\chi \quad \text{and} \quad u_{\gamma\nu}(t) = \int_0^{\infty} \bar{u}_{\gamma\nu}(\chi) e^{i\chi t} d\chi \quad . \quad (27)$$

Whenever necessary, the frequency is set to approach the real axis from below:  $\mathcal{I}m(\chi) \rightarrow 0-$

### 3.4 Should we consider positive frequencies only?

At first glance, the above question appears pointless, as a negative frequency is a mere abstraction, while physical processes go at positive frequencies. Mathematically, a full Fourier decomposition of a *real* field can always be reduced to a decomposition over positive frequencies only.

For example, the full Fourier integral for the stress can be written as

$$\sigma_{\gamma\nu}(t) = \int_{-\infty}^{\infty} \bar{s}_{\gamma\nu}(\omega) e^{i\omega t} d\omega = \int_0^{\infty} \left[ \bar{s}_{\gamma\nu}(\chi) e^{i\chi t} + \bar{s}_{\gamma\nu}(-\chi) e^{-i\chi t} \right] d\chi \quad , \quad (28)$$

where we define  $\chi \equiv |\omega|$ . Denoting complex conjugation with asterisk, we write:

$$\sigma_{\gamma\nu}^*(t) = \int_0^{\infty} \left[ \bar{s}_{\gamma\nu}^*(-\chi) e^{i\chi t} + \bar{s}_{\gamma\nu}^*(\chi) e^{-i\chi t} \right] d\chi \quad . \quad (29)$$

The stress is real:  $\sigma_{\gamma\nu}^*(t) = \sigma_{\gamma\nu}(t)$ . Equating the right-hand sides of (28) and (29), we obtain

$$\bar{s}_{\gamma\nu}(-\chi) = \bar{s}_{\gamma\nu}^*(\chi) \quad , \quad (30)$$

whence

$$\sigma_{\gamma\nu}(t) = \int_0^{\infty} \left[ \bar{s}_{\gamma\nu}(\chi) e^{i\chi t} + \bar{s}_{\gamma\nu}^*(\chi) e^{-i\chi t} \right] d\chi = \mathcal{R}e \int_0^{\infty} 2 \bar{s}_{\gamma\nu}(\chi) e^{i\chi t} d\chi \quad . \quad (31)$$

This leads us to (27), if we set

$$\bar{\sigma}_{\gamma\nu}(\chi) = 2 \bar{s}_{\gamma\nu}(\chi) \quad . \quad (32)$$

While the switch from  $\sigma_{\gamma\nu}(t) = \int_{-\infty}^{\infty} \bar{s}_{\gamma\nu}(\omega) e^{i\omega t} d\omega$  to the expansion  $\sigma_{\gamma\nu}(t) = \int_0^{\infty} \bar{\sigma}_{\gamma\nu}(\omega) e^{i\omega t} d\omega$  makes things simpler, the simplification comes at a cost, as we shall see in a second.

Recall that the tide can be expanded over the modes

$$\omega_{lmpq} \equiv (l-2p)\dot{\omega} + (l-2p+q)\dot{\mathcal{M}} + m(\dot{\Omega} - \dot{\theta}) \approx (l-2p+q)n - m\dot{\theta} \quad , \quad (33)$$

each of which can assume positive or negative values, or be zero. Here  $l, m, p, q$  are some integers,  $\theta$  is the primary's sidereal angle,  $\dot{\theta}$  is its spin rate, while  $\omega, \Omega, \mathcal{M}$  and  $n$  are the secondary's periape, node, mean anomaly, and mean motion. The appropriate tidal frequencies, at which the medium gets loaded, are given by the absolute values of the tidal modes:  $\chi_{lmpq} \equiv |\omega_{lmpq}|$ .

The positively-defined forcing frequencies  $\chi_{lmpq}$  are the actual physical frequency at which the  $lmpq$  term in the expansion for the tidal potential (or stress or strain) oscillates.

The motivation for keeping also the modes  $\omega_{lmpq}$  is subtle: it depends upon the *sign* of  $\omega_{lmpq}$  whether the  $lmpq$  component of the tide lags or advances. Specifically, the phase lag between the  $lmpq$  component of the perturbed primary's potential  $U$  and the  $lmpq$  component of the tide-raising potential  $W$  generated by the secondary is given by

$$\epsilon_{lmpq} = \omega_{lmpq} \Delta t_{lmpq} = |\omega_{lmpq}| \Delta t_{lmpq} \operatorname{sgn} \omega_{lmpq} = \chi_{lmpq} \Delta t_{lmpq} \operatorname{sgn} \omega_{lmpq} \quad , \quad (34)$$

where the time lag  $\Delta t_{lmpq}$  is always positive.

While the lag between the applied stress and resulting strain in a sample of a medium is always positive, the case of tides is more complex: there, the lag can be either positive or negative. This, of course, in no way implies whatever violation of causality (the time lag  $\Delta t_{lmpq}$  is always positive). Rather, this is about the directional difference between the planetocentric positions of the tide-raising body and the resulting bulge. For example, the principal component of the tide,  $lmpq = 2200$ , stays behind (has a positive phase lag  $\epsilon_{2200}$ ) when the secondary is below the synchronous orbit, and advances (has a negative phase lag  $\epsilon_{2200}$ ) when the secondary



is at a higher orbit. To summarise, decomposition of a tide over both positive and negative modes  $\omega_{lmpq}$  (and not just over the positive frequencies  $\chi_{lmpq}$ ) does have a physical meaning, as the sign of a mode  $\omega_{lmpq}$  carries physical information.

Thus we arrive at the following conclusions:

1. As the fields emerging in the tidal theory – the tidal potential, stress, and strain – are all real, their expansions in the frequency domain may, in principle, be written down using the positive frequencies  $\chi$  only.
2. In the tidal theory, the potential (and, consequently, the tidal torque and force) contain components corresponding to the tidal modes  $\omega_{lmpq}$  of both the positive and negative signs. While the  $lmpq$  components of the potential, stress, and strain oscillate at the positive frequencies  $\chi_{lmpq} = |\omega_{lmpq}|$ , the sign of each  $\omega_{lmpq}$  does carry physical information: it distinguishes whether the lagging of the  $lmpq$  component of the bulge is positive or negative (falling behind or advancing). Accordingly, this sign enters explicitly the expression for the appropriate component of the torque or force. Hence a consistent tidal theory should be developed through expansions over both positive and negative tidal modes  $\omega_{lmpq}$  and not just over the positive  $\chi_{lmpq}$ .
3. In order to rewrite the tidal theory in terms of the positively-defined frequencies  $\chi_{lmpq}$  only, one must insert “by hand” the extra multipliers

$$\text{sgn } \omega_{lmpq} = \text{sgn} \left[ (l - 2p + q)n - m\dot{\theta} \right] \quad (35)$$

into the expressions for the  $lmpq$  components of the tidal torque and force.

4. One can employ a rheological law (constitutive equation interconnecting the strain and stress) and a Navier-Stokes equation (the second law of Newton for an element of a viscoelastic medium), to calculate the phase lag  $\epsilon_{lmpq}$  of the primary’s potential  $U_{lmpq}$  relative to the potential  $W_{lmpq}$  generated by the secondary. If both these equations are expanded, in the frequency domain, via positively-defined forcing frequencies  $\chi_{lmpq}$  only, the resulting phase lag, too, will emerge as a function of  $\chi_{lmpq}$ :

$$\epsilon_{lmpq} = \epsilon_l(\chi_{lmpq}) \quad . \quad (36)$$

Within this treatment, one has to equip the lag, “by hand”, with the multiplier (35).

As we saw above, the lag (36) is the argument of the complex Love number  $\bar{k}_l(\chi_{lmpq})$ . Solution of the constitutive and Navier-Stokes equations renders the complex Love numbers, from which one can calculate the lags. Hence the above item [4] may be rephrased in the following manner:

- 4'. Under the convention that  $U_{lmpq} = U(\chi_{lmpq})$  and  $W_{lmpq} = W(\chi_{lmpq})$ , we have:

$$U_{lmpq} = \bar{k}_l(\chi_{lmpq}) W_{lmpq} \quad \text{when } \omega_{lmpq} > 0, \text{ i.e., when } \omega_{lmpq} = \chi_{lmpq} \quad , \quad (37a)$$

$$U_{lmpq} = \bar{k}_l^*(\chi_{lmpq}) W_{lmpq} \quad \text{when } \omega_{lmpq} < 0, \text{ i.e., when } \omega_{lmpq} = -\chi_{lmpq} \quad , \quad (37b)$$

asterisk denoting the complex conjugation.

This ugly convention, a switch from  $\bar{k}_l$  to  $\bar{k}_l^*$ , is the price we pay for employing only the positive frequencies in our expansions, when solving the constitutive and Navier-Stokes equations, to find the Love number. In other words, this is a price for our pretending that  $W_{lmpq}$  and  $U_{lmpq}$  are functions of  $\chi_{lmpq}$  – whereas in reality they are functions of  $\omega_{lmpq}$ .

Alternative to this would be expanding the stress, strain, and the potentials over the positive and negative modes  $\omega_{lmpq}$ , with the negative frequencies showing up in the equations. With the convention that  $U_{lmpq} = U(\omega_{lmpq})$  and  $W_{lmpq} = W(\omega_{lmpq})$ , we would have

$$U_{lmpq} = \bar{k}_l(\omega_{lmpq}) W_{lmpq} \quad , \quad \text{for all } \omega_{lmpq} \quad . \quad (38)$$

All these details can be omitted at the despinning stage, if one keeps only the leading term of the torque and ignores the other terms. Things change, though, when one takes these other terms into account. On crossing of an  $lmpq$  resonance, factor (35) will change its sign. Accordingly, the  $lmpq$  term of the tidal torque (and of the tidal force) will change its sign too.

### 3.5 The complex rigidity and compliance. Stress-strain relaxation

The stress cannot be obtained by means of an integral operator that would map the past history of the strain,  $\mathbb{U}(t')$  over  $t' \in (-\infty, t]$ , to the value of  $\mathbb{S}$  at time  $t$ . The insufficiency of such an operator is evident from the presence of a time-derivative on the right-hand side of (18). Exceptional are the cases of no viscosity (e.g., a purely elastic material).

On the other hand, we expect, on physical grounds, that the operator  $\hat{J}$  inverse to  $\hat{\mu}$  is an integral operator. In other words, we assume that the current value of the strain depends only on the present and past values taken by the stress and not on the current *rate* of change of the stress. This assumption works for weak deformations, i.e., insofar as no plasticity shows up. So we assume that the operator  $\hat{J}$  mapping the stress to the strain is just an integral operator.

Since the forced medium “remembers” the history of loading, the strain at time  $t$  must be a sum of small installments  $\frac{1}{2} J(t - t') d\sigma_{\gamma\nu}(t')$ , each of which stems from a small change  $d\sigma_{\gamma\nu}(t - \tau)$  of the stress at an earlier time  $t' < t$ . The entire history of the past loading results, at the time  $t$ , in a total strain  $u_{\gamma\nu}(t)$  rendered by an integral operator  $\hat{J}(t)$  acting on the entire function  $\sigma_{\gamma\nu}(t')$  and not on its particular value (Karato 2008):

$$2 u_{\gamma\nu}(t) = \hat{J}(t) \sigma_{\gamma\nu} = \int_0^\infty J(\tau) \dot{\sigma}_{\gamma\nu}(t - \tau) d\tau = \int_{-\infty}^t J(t - t') \dot{\sigma}_{\gamma\nu}(t') dt' \quad , \quad (39)$$

where  $t'$  is some earlier time ( $t' < t$ ), overdot denotes  $d/dt'$ , while the “age variable”  $\tau = t - t'$  is reckoned from the current moment  $t$  and is aimed back into the past. The so-defined integral operator  $\hat{J}(t)$  is called the *compliance operator*, while its kernel  $J(t - t')$  goes under the name of the *compliance function* or the *creep-response function*.

Integrating (39) by parts, we recast the compliance operator into the form of

$$2 u_{\gamma\nu}(t) = \hat{J}(t) \sigma_{\gamma\nu} = J(0) \sigma_{\gamma\nu}(t) - J(\infty) \sigma_{\gamma\nu}(-\infty) + \int_0^\infty \dot{J}(\tau) \sigma_{\gamma\nu}(t - \tau) d\tau \quad (40a)$$

$$= J(0) \sigma_{\gamma\nu}(t) - J(\infty) \sigma_{\gamma\nu}(-\infty) + \int_{-\infty}^t \dot{J}(t - t') \sigma_{\gamma\nu}(t') dt' \quad . \quad (40b)$$

The quantity  $J(\infty)$  is the *relaxed compliance*. Being the asymptotic value of  $J(t - t')$  at  $t - t' \rightarrow \infty$ , this parameter corresponds to the strain after complete relaxation. The load in the infinite past may be assumed zero, and the term  $- J(\infty) \sigma_{\gamma\nu}(-\infty)$  may be dropped

The second important quantity emerging in (40) is the *unrelaxed compliance*  $J(0)$ , which is the value of the compliance function  $J(t - t')$  at  $t - t' = 0$ . This parameter describes the instantaneous reaction to stressing, and thus defines the *elastic* part of the deformation (the rest of the deformation being viscous and hereditary). Thus the term containing the unrelaxed compliance  $J(0)$  should be kept. The term, though, can be absorbed into the integral if we agree that the elastic contribution enters the compliance function not as <sup>4</sup>

$$J(t - t') = J(0) + \text{viscous and hereditary terms} \quad , \quad (41)$$

but as

$$J(t - t') = J(0) \Theta(t - t') + \text{viscous and hereditary terms} \quad , \quad (42)$$

the Heaviside step-function  $\Theta(t - t')$  being unity for  $t - t' \geq 0$ , and zero for  $t - t' < 0$ . As the derivative of the step-function is the delta-function  $\delta(t - t')$ , we can write (40b) simply as

$$2 u_{\gamma\nu}(t) = \hat{J}(t) \sigma_{\gamma\nu} = \int_{-\infty}^t \dot{J}(t - t') \sigma_{\gamma\nu}(t') dt' \quad , \quad \text{with } J(t - t') \text{ containing } J(0) \Theta(t - t') \quad . \quad (43)$$

Equations (39), (40), (43) are but different expressions for the compliance operator  $\hat{J}$  acting as

$$2 u_{\gamma\nu} = \hat{J} \sigma_{\gamma\nu} \quad . \quad (44)$$

Inverse to the compliance operator is the rigidity operator  $\hat{\mu}$  defined through

$$\sigma_{\gamma\nu} = 2 \hat{\mu} u_{\gamma\nu} \quad . \quad (45)$$

Generally,  $\hat{\mu}$  is not just an integral operator, but is an integro-differential operator. So it cannot take the form of  $\sigma_{\gamma\nu}(t) = 2 \int_{-\infty}^t \dot{\mu}(t - t') u_{\gamma\nu}(t') dt'$ . However it can be written as

$$\sigma_{\gamma\nu}(t) = 2 \int_{-\infty}^t \mu(t - t') \dot{u}_{\gamma\nu}(t') dt' \quad , \quad (46)$$

if we permit the kernel  $\mu(t - t')$  to contain a term  $\eta \delta(t - t')$ , where  $\delta(t - t')$  is the delta-function. After integration, this term will furnish the viscous part of the stress,  $2 \eta \dot{u}_{\gamma\nu}$ .

The kernel  $\mu(t - t')$  goes under the name of the *stress-relaxation function*. Its time-independent part is  $\mu(0) \Theta(t - t')$ , where the *unrelaxed rigidity*  $\mu(0)$  is inverse to the unrelaxed compliance  $J(0)$  and describes the elastic part of deformation. Each term in  $\mu(t - t')$ , which neither is a constant nor contains a delta-function, is responsible for hereditary reaction.

For more details on the stress-strain relaxation formalism see the book by Karato (2008).

### 3.6 Stress-strain relaxation in the frequency domain

Let us introduce the complex compliance  $\bar{J}(\chi)$  and the complex rigidity  $\bar{\mu}(\chi)$ , which are, by definition, the Fourier images **not** of the  $J(\tau)$  and  $\mu(\tau)$  functions, but of their time-derivatives:<sup>5</sup>

$$\int_0^\infty \bar{J}(\chi) e^{i\chi\tau} d\chi = \dot{J}(\tau) \quad , \quad \text{where} \quad \bar{J}(\chi) = \int_0^\infty \dot{J}(\tau) e^{-i\chi\tau} d\tau \quad . \quad (47)$$

---

<sup>4</sup> Expressing the stress through the strain, we encountered three possibilities: the elastic stress was simply proportional to the strain, the viscous stress was proportional to the time-derivative of the strain, while the hereditary stress was expressed by an integral operator  $\hat{\mu}$ . However, when we express the strain through the stress, we place the viscosity into the integral operator, so the purely viscous reaction also looks like hereditary. It is our convention, though, to apply the term *hereditary* to delayed reactions *other than purely viscous*.

<sup>5</sup> Recall that it is the time-derivative of  $J(\tau)$  that is the kernel of the integral operator (43). Hence, to arrive at (50), we have to define  $\bar{J}(\chi)$  as the Fourier image of  $\dot{J}(\tau)$ .

and

$$\int_0^\infty \bar{\mu}(\chi) e^{i\chi\tau} d\chi = \dot{\mu}(\tau) \quad , \quad \text{where} \quad \bar{\mu}(\chi) = \int_0^\infty \dot{\mu}(\tau) e^{-i\chi\tau} d\tau \quad , \quad (48)$$

the integrations over  $\tau$  spanning the interval  $[0, \infty)$ , as both kernels are nil for  $\tau < 0$  anyway. In (47) and (48), we made use of the fact (explained in subsection 3.4) that, when expanding real fields, it is sufficient to use only positive frequencies.

Expression (39), in combination with the Fourier expansions (27) and with (47), furnishes:

$$2 \int_0^\infty \bar{u}_{\gamma\nu}(\chi) e^{i\chi t} d\chi = \int_0^\infty \bar{\sigma}_{\mu\nu}(\chi) \bar{J}(\chi) e^{i\chi t} d\chi \quad , \quad (49)$$

which leads us to:

$$2 \bar{u}_{\gamma\nu}(\chi) = \bar{J}(\chi) \bar{\sigma}_{\gamma\nu}(\chi) \quad . \quad (50)$$

Similarly, insertion of (27) into (46) leads to the relation

$$\bar{\sigma}_{\gamma\nu}(\chi) = 2 \bar{\mu}(\chi) \bar{u}_{\gamma\nu}(\chi) \quad , \quad (51)$$

comparison whereof with (50) immediately entails:

$$\bar{J}(\chi) \bar{\mu}(\chi) = 1 \quad . \quad (52)$$

Writing down the complex rigidity and compliance as

$$\bar{\mu}(\chi) = |\bar{\mu}(\chi)| \exp[i\delta(\chi)] \quad (53)$$

and

$$\bar{J}(\chi) = |\bar{J}(\chi)| \exp[-i\delta(\chi)] \quad , \quad (54)$$

we split (52) into two expressions:

$$|\bar{J}(\chi)| = \frac{1}{|\bar{\mu}(\chi)|} \quad (55)$$

and

$$\varphi_u(\chi) = \varphi_\sigma(\chi) - \delta(\chi) \quad . \quad (56)$$

From the latter, we see that the angle  $\delta(\chi)$  is a measure of lagging of a strain harmonic mode relative to the appropriate harmonic mode of the stress. It is evident from (53 - 54) that

$$\tan \delta(\chi) \equiv - \frac{\text{Im} [\bar{J}(\chi)]}{\text{Re} [\bar{J}(\chi)]} = \frac{\text{Im} [\bar{\mu}(\chi)]}{\text{Re} [\bar{\mu}(\chi)]} \quad . \quad (57)$$

## 4 Complex Love numbers

The developments presented in this section will rest on a very important theorem from solid-state mechanics. The theorem, known as the *correspondence principle*, also goes under the name of *elastic-viscoelastic analogy*. The theorem applies to linear deformations in the absence of nonconservative (inertial) forces. While the literature attributes the authorship of the theorem to different scholars, its true pioneer was Sir George Darwin (1879). One of the corollaries ensuing from this theorem is that, in the frequency domain, the complex Love numbers are expressed via

the complex rigidity or compliance in the same way as the static Love numbers are expressed via the relaxed rigidity or compliance.

As was pointed out much later by Biot (1954, 1958), the theorem is inapplicable to non-potential forces. Hence the said corollary fails in the case of librating bodies, because of the presence of the inertial force<sup>6</sup>  $-\dot{\vec{\omega}} \times \vec{r}\rho$ , where  $\rho$  is the density and  $\vec{\omega}$  is the libration angular velocity. So the standard expression (3) for the Love numbers, generally, cannot be employed for librating bodies.

Subsection 4.1 below explains the transition from the stationary Love numbers to their dynamical counterparts, the so-called Love operators. We present this formalism in the frequency domain, in the spirit of Zahn (1966) who pioneered this approach in application to a purely viscous medium. Subsection 4.2 addresses the negative tidal modes emerging in the Darwin-Kaula expansion for tides. Employing the correspondence principle, in subsection 4.3 we then write down the expressions for the factors  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) = -\mathcal{I}m[\bar{k}_l(\chi)]$  emerging in the expansion for tides. Some technical details of this derivation are discussed in subsections 4.4 - 4.5.

For more on the correspondence principle and its applicability to Phobos see Appendix D.

## 4.1 From the Love numbers to the Love operators

A homogeneous incompressible primary, when perturbed by a static secondary, yields its form and, consequently, has its potential changed. The  $l^{th}$  spherical harmonic  $U_l(\vec{r})$  of the resulting increment of the primary's exterior potential is related to the  $l^{th}$  spherical harmonic  $W_l(\vec{R}, \vec{r})$  of the perturbing exterior potential through (2).

As the realistic disturbances are never static (except for synchronous orbiting), the Love numbers become operators:

$$U_l(\vec{r}, t) = \left(\frac{R}{r}\right)^{l+1} \hat{k}_l(t) W_l(\vec{R}, \vec{r}^*, t') \quad . \quad (58)$$

A Love operator acts neither on the value of  $W$  at the current time  $t$ , nor at its value at an earlier time  $t'$ , but acts on the entire shape of the function  $W_l(\vec{R}, \vec{r}^*, t')$ , with  $t'$  belonging to the semi-interval  $(-\infty, t)$ . This is why we prefer to write  $\hat{k}_l(t)$  and not  $\hat{k}_l(t, t')$ .

Being linear for weak forcing, the operators must read:

$$U_l(\vec{r}, t) = \left(\frac{R}{r}\right)^{l+1} \int_{\tau=0}^{\tau=\infty} k_l(\tau) \dot{W}_l(\vec{R}, \vec{r}^*, t - \tau) d\tau = \left(\frac{R}{r}\right)^{l+1} \int_{t'=-\infty}^{t'=t} k_l(t - t') \dot{W}_l(\vec{R}, \vec{r}^*, t') dt' \quad (59a)$$

or, after integration by parts:

$$U_l(\vec{r}, t) = \left(\frac{R}{r}\right)^{l+1} [k_l(0)W(t) - k_l(\infty)W(-\infty)] + \left(\frac{R}{r}\right)^{l+1} \int_0^\infty \dot{k}_l(\tau) W_l(\vec{R}, \vec{r}^*, t - \tau) d\tau \quad (59b)$$

$$= \left(\frac{R}{r}\right)^{l+1} [k_l(0)W(t) - k_l(\infty)W(-\infty)] + \left(\frac{R}{r}\right)^{l+1} \int_{-\infty}^t \dot{k}_l(t - t') W_l(\vec{R}, \vec{r}^*, t') dt' \quad (59c)$$

$$= - \left(\frac{R}{r}\right)^{l+1} k_l(\infty)W(-\infty) + \left(\frac{R}{r}\right)^{l+1} \int_{-\infty}^t \frac{d}{dt} [k_l(t - t') - k_l(0) + k_l(0)\Theta(t - t')] W_l(\vec{R}, \vec{r}^*, t') dt' \quad . \quad (59d)$$

---

<sup>6</sup> The centripetal term is potential and causes no troubles, except for the necessity to introduce a degree-0 Love number.

Just as in the case of the compliance operator (39 - 40), in expressions (59) we obtain the terms  $k_l(0)W(t)$  and  $-k_l(\infty)W(-\infty)$ . Of the latter term, we can get rid by setting  $W(-\infty)$  nil, while the former term may be incorporated into the kernel in exactly the same way as in (41 - 43). Thus, dropping the unphysical term with  $W(-\infty)$ , and inserting the elastic term into the Love number not as  $k_l(0)$  but as  $k_l(0)\Theta(t-t')$ , we simplify (59d) to

$$U_l(\vec{r}, t) = \left(\frac{R}{r}\right)^{l+1} \int_{-\infty}^t \dot{k}_l(t-t') W_l(\vec{R}, \vec{r}^*, t') dt', \quad (60)$$

with  $k_l(t-t')$  now including, as its part,  $k_l(0)\Theta(t-t')$  instead of  $k_l(0)$ .

Were the body perfectly elastic,  $k_l(t-t')$  would consist of the instantaneous-reaction term  $k_l(0)\Theta(t-t')$  only. Accordingly, the time-derivative of  $k_l$  would be:  $\dot{k}_l(t-t') = k_l\delta(t-t')$  where  $k_l \equiv k_l(0)$ , so expressions (59 - 60) would coincide with (2).

Similarly to introducing the complex compliance, one can define the complex Love numbers as Fourier transforms of  $\dot{k}_l(\tau)$ :

$$\int_0^\infty \bar{k}_l(\chi) e^{i\chi\tau} d\chi = \dot{k}_l(\tau) \quad , \quad (61)$$

the overdot standing for  $d/d\tau$ . Churkin (1998) suggested to term the time-derivatives  $\dot{k}_l(t)$  as the *Love functions*.<sup>7</sup> Inversion of (61) trivially yields:

$$\bar{k}_l(\chi) = \int_0^\infty \dot{k}_l(\tau) e^{-i\chi\tau} d\tau = k_l(0) + i\chi \int_0^\infty [k_l(\tau) - k_l(0)\Theta(\tau)] e^{-i\chi\tau} d\tau \quad , \quad (62)$$

where we integrated only from 0 because the future disturbance contributes nothing to the present distortion, so  $k_l(\tau)$  vanishes at  $\tau < 0$ . Recall that the time  $\tau$  denotes the difference  $t - t'$ . So  $\tau$  is reckoned from the present moment  $t$  and is directed back into the past.

Defining in the standard manner the Fourier components  $\bar{U}_l(\chi)$  and  $\bar{W}_l(\chi)$  of functions  $U_l(t)$  and  $W_l(t)$ , we write (59) in the frequency domain:

$$\bar{U}_l(\chi) = \left(\frac{R}{r}\right)^{l+1} \bar{k}_l(\chi) \bar{W}_l(\chi) \quad , \quad (63)$$

where we denote the frequency simply by  $\chi$  instead of the awkward  $\chi_{lmpq}$ . To employ (63) in the tidal theory, one has to know the frequency-dependencies  $\bar{k}_l(\chi)$ .

## 4.2 The positive forcing frequencies $\chi \equiv |\omega|$ vs. the positive and negative tidal modes $\omega$

It should be remembered that, by relying on formula (63), we place ourselves on thin ice, because the similarity of this formula to (50) and (51) is deceptive.

In (50) and (51), it was legitimate to limit our expansions of the stress and the strain to positive frequencies  $\chi$  only. Had we carried out those expansions over both positive and negative frequencies  $\omega$ , we would have obtained, instead of (50) and (51), similar expressions

$$2\bar{u}_{\gamma\nu}(\omega) = \bar{J}(\omega) \bar{\sigma}_{\gamma\nu}(\omega) \quad \text{and} \quad \bar{\sigma}_{\gamma\nu}(\omega) = 2\bar{\mu}(\omega) \bar{u}_{\gamma\nu}(\omega) \quad . \quad (64)$$

For positive  $\omega$ , these would simply coincide with (50) and (51), if we rename  $\omega$  as  $\chi$ . For negative  $\omega = -\chi$ , the resulting expressions would read as

$$2\bar{u}_{\gamma\nu}(-\chi) = \bar{J}(-\chi) \bar{\sigma}_{\gamma\nu}(-\chi) \quad \text{and} \quad \bar{\sigma}_{\gamma\nu}(-\chi) = 2\bar{\mu}(-\chi) \bar{u}_{\gamma\nu}(-\chi) \quad , \quad (65)$$

---

<sup>7</sup> Churkin (1998) used functions which he called  $k_l(t)$  and which were, due to a difference in notations, the same as our  $\dot{k}_l(\tau)$ .

where we stick to the agreement that  $\chi$  always stands for a positive quantity. In accordance with (30), complex conjugation of (65) would then return us to (64).

Physically, the negative-frequency components of the stress or strain are nonexistent. If brought into consideration, they are obliged to obey (30) and, thus, should play no role, except for a harmless renormalisation of the Fourier components in (32).

When we say that the physically measurable stress  $\sigma_{\gamma\nu}(t)$  is equal to  $\sum \mathcal{R}e \left[ \bar{\sigma}_{\gamma\nu}(\chi) e^{i\chi t} \right]$ , it is unimportant to us whether the  $\chi$ -contribution in  $\sigma_{\gamma\nu}(t)$  comes from the term  $\bar{\sigma}_{\gamma\nu}(\chi) e^{i\chi t}$  only, or also from the term  $\bar{\sigma}_{\gamma\nu}(-\chi) e^{i(-\chi)t}$ . Indeed, the real part of the latter is a clone of the real part of the former (and it is only the former term that is physical). However, things remain that simple only for the stress and the strain.

As we emphasised in subsection 3.4, the situation with the potentials is drastically different. While the physically measurable potential  $U(t)$  is still equal to  $\sum \mathcal{R}e \left[ \bar{U}(\chi) e^{i\chi t} \right]$ , it is now *important* to distinguish whether the  $\chi$ -contribution in  $U(t)$  comes from the term  $\bar{U}_{\gamma\nu}(\chi) e^{i\chi t}$  or from the term  $\bar{U}(-\chi) e^{i(-\chi)t}$ , or perhaps from both. Although the negative mode  $-\chi$  would bring the same input as the positive mode  $\chi$ , these inputs will contribute differently into the tidal torque. As can be seen from (285), the secular part of the tidal torque is proportional to  $\sin \epsilon_l$ , where  $\epsilon_l \equiv \omega_{lmpq} \Delta t_{lmpq}$ , with the time lag  $\Delta t_{lmpq}$  being positively defined – see formula (109). Thus the secular part of the tidal torque explicitly contains the sign of the tidal mode  $\omega_{lmpq}$ .

For this reason, as explained in subsection 3.4, a more accurate form of formula (63) should be:

$$\bar{U}_l(\omega) = \bar{k}_l(\omega) \bar{W}_l(\omega) \quad , \quad (66)$$

where  $\omega$  can be of any sign.

If however, we pretend that the potentials depend on the physical frequency  $\chi = |\omega|$  only, i.e., if we always write  $U(\omega)$  as  $U(\chi)$ , then (63) must be written as:

$$\bar{U}_l(\chi) = \bar{k}_l(\chi) \bar{W}_l(\chi) \quad , \quad \text{when } \chi = |\omega| \quad \text{for } \omega > 0 \quad , \quad (67a)$$

and

$$\bar{U}_l(\chi) = \bar{k}_l^*(\chi) \bar{W}_l(\chi) \quad , \quad \text{when } \chi = |\omega| \quad \text{for } \omega < 0 \quad . \quad (67b)$$

Unless we keep this detail in mind, we shall get a wrong sign for the  $lmpq$  component of the torque after the despinning secondary crosses the appropriate commensurability. (We shall, of course, be able to mend this by simply inserting the sign  $\text{sgn } \omega_{lmpq}$  by hand.)

### 4.3 The complex Love number as a function of the complex compliance

While the static Love numbers depend on the static rigidity modulus  $\mu$  via (3), it is not readily apparent that the same relation interconnects  $\bar{k}_l(\chi)$  with  $\bar{\mu}(\chi)$ , the quantities that are the Fourier components of the time-derivatives of  $k_2(t')$  and  $\mu(t')$ . Fortunately, the *correspondence principle* (discussed in Appendix D) tells us that, in many situations, the viscoelastic operational moduli  $\bar{\mu}(\chi)$  or  $\bar{J}(\chi)$  obey the same algebraic relations as the elastic parameters  $\mu$  or  $J$ . This is why, in these situations, the Fourier or Laplace transform of our viscoelastic equations will mimic (228a - 228b), except that all the functions will acquire overbars:  $\bar{\sigma}_{\gamma\nu} = 2 \bar{\mu} \bar{u}_{\gamma\nu}$ , etc.

So their solution, too, will be  $\bar{U}_l = \bar{k}_l \bar{W}_l$ , with  $\bar{k}_l$  retaining the same functional dependence on  $\rho$ ,  $R$ , and  $\bar{\mu}$  as in (3), except that now  $\mu$  will have an overbar:

$$\begin{aligned} \bar{k}_l(\chi) &= \frac{3}{2(l-1)} \frac{1}{1 + \frac{(2l^2 + 4l + 3)\bar{\mu}(\chi)}{lg\rho R}} = \frac{3}{2(l-1)} \frac{1}{1 + A_l \bar{\mu}(\chi)/\mu} \\ &= \frac{3}{2(l-1)} \frac{1}{1 + A_l J/\bar{J}(\chi)} = \frac{3}{2(l-1)} \frac{\bar{J}(\chi)}{\bar{J}(\chi) + A_l J} \end{aligned} \quad (68)$$

Here the coefficients  $A_l$  are defined via the unrelaxed quantities  $\mu = \mu(0) = 1/J = 1/J(0)$  in the same manner as the static  $A_l$  were introduced through the static (relaxed)  $\mu = 1/J$  in formulae (3).

The moral of the story is that, at low frequencies, each  $\bar{k}_l$  depends upon  $\bar{\mu}$  (or upon  $\bar{J}$ ) in the same way as its static counterpart  $k_l$  depends upon the static  $\mu$  (or upon the static  $J$ ). This happens, because at low frequencies we neglect the acceleration term in the equation of motion (231b), so this equation still looks like (228b).

Representing a complex Love number as

$$\bar{k}_l(\chi) = \mathcal{R}e [\bar{k}_l(\chi)] + i \mathcal{I}m [\bar{k}_l(\chi)] = |\bar{k}_l(\chi)| e^{-i\epsilon_l(\chi)} \quad (69)$$

we can write for the phase lag  $\epsilon_l(\chi)$ :

$$\tan \epsilon_l(\chi) \equiv - \frac{\mathcal{I}m [\bar{k}_l(\chi)]}{\mathcal{R}e [\bar{k}_l(\chi)]} \quad (70)$$

or, equivalently:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) = - \mathcal{I}m [\bar{k}_l(\chi)] \quad . \quad (71)$$

The products  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$  standing on the left-hand side in (71) emerge also in the Fourier series for the tidal potential. Therefore it is these products (and not  $k_l/Q$ ) that should enter the expansions for forces, torques, and the damping rate. This is the link between the body's rheology and the history of its spin: from  $\bar{J}(\chi)$  to  $\bar{k}_l(\chi)$  to  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$ , the latter being employed in the theory of bodily tides.

Through simple algebra, expressions (68) entail:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) = - \mathcal{I}m [\bar{k}_l(\chi)] = \frac{3}{2(l-1)} \frac{-A_l J \mathcal{I}m [\bar{J}(\chi)]}{(\mathcal{R}e [\bar{J}(\chi)] + A_l J)^2 + (\mathcal{I}m [\bar{J}(\chi)])^2} \quad . \quad (72)$$

As we know from subsections 3.4 and 4.2, formulae (70 - 72) should be used with care. Since in reality the potential  $\bar{U}$  and therefore also  $\bar{k}_l$  are functions not of  $\chi$  but of  $\omega$ , then formulae (72) should be equipped with multipliers  $\text{sgn } \omega_{lmpq}$ , when plugged into the expression for the  $lmpq$  component of the tidal force or torque. This prescription is equivalent to (67).

#### 4.4 Should we write $\bar{k}_{lmpq}$ and $\epsilon_{lmpq}$ , or would $\bar{k}_l$ and $\epsilon_l$ be enough?

In the preceding subsection, the static relation (2) was generalised to evolving settings as

$$U_{lmpq}(\vec{r}, t) = \left( \frac{R}{r} \right)^{l+1} \hat{k}_l(t) W_{lmpq}(\vec{R}, \vec{r}^*, t') \quad , \quad (73)$$



where  $lmpq$  is a quadruple of integers employed to number a Fourier mode in the Darwin-Kaula expansion (100) of the tide, while  $U_{lmpq}(\vec{r}, t)$  and  $W_{lmpq}(\vec{R}, \vec{r}^*, t')$  are the harmonics containing  $\cos(\chi_{lmpq}t - \epsilon_{lmpq})$  and  $\cos(\chi_{lmpq}t')$  correspondingly.

One might be tempted to generalise (2) even further to

$$U_{lmpq}(\vec{r}, t) = \left( \frac{R}{r} \right)^{l+1} \hat{k}_{lmpq}(t) W_{lmpq}(\vec{R}, \vec{r}^*, t') ,$$

with the Love operator (and, consequently, its kernel, the Love function) bearing dependence upon  $m$ ,  $p$ , and  $q$ . Accordingly, (63) would become

$$\bar{U}_{lmpq}(\chi) = \bar{k}_{lmpq}(\chi) \bar{W}_{lmpq}(\chi) . \quad (74)$$

Fortunately, insofar as the Correspondence Principle is valid, the functional form of the function  $\bar{k}_{lmpq}(\chi)$  depends upon  $l$  only and, thus, can be written down simply as  $\bar{k}_l(\chi_{lmpq})$ . We know this from the considerations offered after equations (228a - 228b). There we explained that  $\bar{k}_l$  depends on  $\chi = \chi_{lmpq}$  only via  $\bar{J}(\chi)$ , while the functional form of  $\bar{k}_l$  bears no dependence on  $\chi = \chi_{lmpq}$  and, therefore, no dependence on  $m$ ,  $p$ ,  $q$ .

The phase lag is often denoted as  $\epsilon_{lmpq}$ , a time-honoured tradition established by Kaula (1964). However, as the lag is expressed through  $\bar{k}_l$  via (70), we see that all said above about  $\bar{k}_l$  applies to the lag too: while the functional form of the dependency  $\epsilon_{lmpq}(\chi)$  may be different for different  $ls$ , it is invariant under the other three integers, so the notation  $\epsilon_l(\chi_{lmpq})$  would be more adequate.

It should be mentioned, though, that for bodies of pronounced non-sphericity coupling between the spherical harmonics furnishes the Love numbers and lags whose expressions through the frequency, for a fixed  $l$ , have different functional forms for different  $m$ ,  $p$ ,  $q$ . In these cases, the notations  $\bar{k}_{lmpq}$  and  $\epsilon_{lmpq}$  become necessary (Smith 1974; Wahr 1981a,b,c; Dehant 1987a,b). For a slightly non-spherical body, the Love numbers differ from the Love numbers of the spherical reference body by a term of the order of the flattening, so a small non-sphericity can usually be neglected.

## 4.5 Rigidity vs self-gravitation

For small bodies and small terrestrial planets, the values of  $A_l$  vary from about unity to dozens to hundreds. For example,  $A_2$  is about 2 for the Earth (Efroimsky 2012), about 20 for Mars (Efroimsky & Lainey 2007), about 80 for the Moon (Efroimsky 2012), and about 200 for Iapetus (Castillo-Rogez et al. 2011). For superearths, the values will be much smaller than unity, though.

Insofar as

$$A_l \frac{J}{|\bar{J}(\chi)|} \gg 1 , \quad (75)$$

one can approximate (68) with

$$\bar{k}_l(\chi) = - \frac{3}{2(l-1)} \frac{\bar{J}(\chi)}{\bar{J}(\chi) + A_l J} = - \frac{3}{2} \frac{\bar{J}(\chi)}{A_l J} + O\left(|\bar{J}/(A_l J)|^2\right) , \quad (76)$$

except in the closest vicinity of an  $lmpq$  resonance, where the tidal frequency  $\chi_{lmpq}$  approaches nil, and  $\bar{J}$  diverges for some rheologies – like, for example, for those of Maxwell or Andrade.

Whenever the approximate formula (76) is applicable, we can rewrite (70) as

$$\tan \epsilon(\chi) \equiv - \frac{\mathcal{I}m[\bar{k}_l(\chi)]}{\mathcal{R}e[\bar{k}_l(\chi)]} \approx - \frac{\mathcal{I}m[\bar{J}(\chi)]}{\mathcal{R}e[\bar{J}(\chi)]} = \tan \delta(\chi) , \quad (77)$$

wherefrom we readily deduce that the phase lag  $\epsilon(\chi)$  of the tidal frequency  $\chi$  coincides with the phase lag of the complex compliance:

$$\epsilon(\chi) \approx \delta(\chi) \quad , \quad (78)$$

provided  $\chi$  is not too close to nil (i.e., provided we are not too close to the commensurability). This way, insofar as the condition (71) is fulfilled, the component  $\bar{U}_l(\chi)$  of the primary's potential lags behind the component  $\bar{W}_l(\chi)$  of the perturbed potential by the same phase angle as the strain lags behind the stress at frequency  $\chi$  in a sample of the material. Dependent upon the rheology, a vanishing tidal frequency may or may not limit the applicability of (71) and thus cause a considerable difference between  $\epsilon$  and  $\delta$ .

In other words, the suggested approximation is valid insofar as changes of shape are determined solely by the local material properties, and not by self-gravitation of the object as a whole. Whether this is so or not – depends upon the rheological model. For a Voigt or SAS<sup>8</sup> solid in the limit of  $\chi \rightarrow 0$ , we have  $\bar{J}(\chi) \rightarrow J$ , so the zero-frequency limit of  $\bar{k}_l(\chi)$  is the static Love number  $k_l \equiv |\bar{k}(0)|$ . In this case, approximation (76 - 78) remains applicable all the way down to  $\chi = 0$ . For the Maxwell and Andrade models, however, one obtains, for vanishing frequency:  $\bar{J}(\chi) \sim 1/(\eta\chi)$ , whence  $\bar{\mu} \sim \eta\chi$  and  $\bar{k}_2(\chi)$  approaches the hydrodynamical Love number  $k_2^{(hyd)} = 3/2$ .

We see that, for the Voigt and SAS models, approximation (78) can work, for  $A_l \gg 1$ , at all frequencies, because the condition  $A_L \gg 1$  can be set for all frequencies. For the Maxwell and Andrade solids, this condition holds only at frequencies larger than  $\tau_M^{-1} A_l^{-1} = \frac{\mu}{\eta} A_l^{-1}$ , and so does the approximation (78). Indeed, at frequencies below this threshold, self-gravitation “beats” the local material properties of the body, and the behaviour of the tidal lag deviates from that of the lag in a sample. This deviation will be indicated more clearly by formula (94) in the next section. The fact that, for some models, the tidal lag  $\epsilon$  deviates from the material lag angle  $\delta$  at the lowest frequencies should be kept in mind when one wants to explore *crossing* of a resonance.

A standard caveat is in order, concerning formulae (76 - 78). Since in reality the potential  $\bar{U}$  is a function of  $\omega$  and not  $\chi$ , our illegitimate use of  $\chi$  should be compensated by multiplying the function  $\epsilon_l(\chi_{lmpq})$  with  $\text{sgn} \omega_{lmpq}$ , when the lag shows up in the expression for the tidal force or torque.

## 4.6 The case of inhomogeneous bodies

Tidal dissipation within a multilayer near-spherical body is studied through expanding the involved fields over the spherical harmonics in each layer, setting the boundary conditions on the outer surface, and using the matching conditions on boundaries between layers. This formalism was developed by Alterman et al (1959). An updated discussion of the method can be found in Sabadini & Vermeersen (2004). For a brief review, see Legros et al (2006).

Calculation of tidal dissipation in a Jovian planet is an even more formidable task (see Remus et al. 2012a and references therein). However dissipation in a giant planet with a solid core may turn out to be approachable by analytic means (Remus et al. 2011, 2012b).

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<sup>8</sup> The acronym *SAS* stands for the *Standard Anelastic Solid*, which is another name for the Hohenemser-Prager viscoelastic model. See the Appendix for details.

## 5 Dissipation at different frequencies

### 5.1 The data collected on the Earth: in the lab, over seismological basins, and through geodetic measurements

In Efroimsky & Lainey (2007), we considered the generic rheological model

$$Q = (\mathcal{E} \chi)^\alpha, \quad (79a)$$

where  $\chi$  is the tidal frequency and  $\mathcal{E}$  is a parameter having the dimensions of time. The physical meaning of this parameter is elucidated in *Ibid.*. Under the special choice of  $\alpha = -1$  and for sufficiently large values of  $Q$ , this parameter coincides with the time lag  $\Delta t$  which, for this special rheology, turns out to be the same at all frequencies.

Actual experiments register not the inverse quality factor but the phase lag between the reaction and the action. So the empirical law should rather be written down as

$$\frac{1}{\sin \delta} = (\mathcal{E} \chi)^\alpha, \quad (79b)$$

which is equivalent to (79a), provided the  $Q$  factor is defined there as  $Q_{energy}$  and not as  $Q_{work}$  – see subsection 2.2 for details.

The applicability realm of the empirical power law (79) is remarkably broad – in terms of both the physical constituency of the bodies and their chemical composition. Most intriguing is the robust universality of the values taken by the index  $\alpha$  for very different materials: between 0.2 and 0.4 for ices and silicates, and between 0.14 and 0.2 for partial melts. Historically, two communities independently converged on this form of dependence.

In the material sciences, the rheological model (86), wherefrom the power law (79b) stems, traces its lineage to the groundbreaking work by Andrade (1910) who explored creep in metals. Through the subsequent century, this law was found to be applicable to a vast variety of other materials, including minerals (Weertman & Weertman 1975, Tan et al. 1997) and their partial melts (Fontaine et al. 2005). As recently discovered by McCarthy et al. (2007) and Castillo-Rogez (2009), the same law, with almost the same values of  $\alpha$ , also applies to ices. The result is milestone, taken the physical and chemical differences between ices and silicates. It is agreed upon that in crystalline materials the Andrade regime can find its microscopic origin both in the dynamics of dislocations (Karato & Spetzler 1990) and in the grain-boundary diffusional creep (Gribb & Cooper 1998). As the same behaviour is inherent in metals, silicates, ices, and even glass-polyester composites (Nechada et al. 2005), it should stem from a single underlying phenomenon determined by some principles more general than specific material properties. An attempt to find such a universal mechanism was undertaken by Miguel et al. (2002). See also the theoretical considerations offered in Karato & Spetzler (1990).

In seismology, the power law (79) became popular in the second part of the XX<sup>th</sup> century, with the progress of precise measurements on large seismological basins (Mitchell 1995, Stachnik et al. 2004, Shito et al. 2004). Further confirmation of this law came from geodetic experiments that included: (a) satellite laser ranging (SLR) measurements of tidal variations in the  $J_2$  component of the gravity field of the Earth; (b) space-based observations of tidal variations in the Earth's rotation rate; and (c) space-based measurements of the Chandler Wobble period and damping (Benjamin et al. 2006, Eanes & Bettadpur 1996, Eanes 1995). Not surprisingly, the Andrade law became a key element in the recent attempt to construct a universal rheological model of the Earth's mantle (Birger 2007). This law also became a component of the non-hydrostatic-equilibrium model for the zonal tides in an inelastic Earth by Defraigne & Smits (1999), a model that became the basis for the IERS Conventions (Petit & Luzum 2010). While the lab experiments give for  $\alpha$  values within 0.2 – 0.4, the geodetic techniques favour the interval

0.14 – 0.2. This minor discrepancy may have emerged due to the presence of partial melt in the mantle and, possibly, due to nonlinearity at high bounding pressures in the lower mantle. The universality of the Andrade law compels us to assume that (79) works equally well for other terrestrial bodies. Similarly, the applicability of (79) to samples of ices in the lab is likely to indicate that this law can be employed for description of an icy moon as a whole.

Karato & Spetzler (1990) argue that at frequencies below a certain threshold  $\chi_0$  anelasticity gives way to purely viscoelastic behaviour, so the parameter  $\alpha$  becomes close to unity.<sup>9</sup> For the Earth’s mantle, the threshold corresponds to the time-scale about a year or slightly longer. Although in Karato & Spetzler (1990) the rheological law is written in terms of  $1/Q$ , we shall substitute it with a law more appropriate to the studies of tides:

$$k_l \sin \epsilon_l = (\mathcal{E} \chi)^{-p}, \quad \text{where} \quad \begin{array}{ll} p = 0.2 - 0.4 & \text{for } \chi > \chi_0 \\ \text{and} & p \sim 1 \quad \text{for } \chi < \chi_0 \end{array}, \quad (80)$$

$\chi$  being the frequency, and  $\chi_0$  being the frequency threshold below which viscosity takes over anelasticity.

The reason why we write the power scaling law as (80) and not as (79) is that at the lowest frequencies the geodetic measurements give us actually  $k_l \sin \epsilon_l = -\mathcal{I}m [\bar{k}_l(\chi)]$  and not the lag angle  $\delta$  in a sample (e.g., Benjamin et al. 2006). For this same reason, we denoted the exponents in (79) and (80) with different letters,  $\alpha$  and  $p$ . Below we shall see that these exponents do not always coincide. Another reason for giving preference to (80) is that not only the sine of the lag but also the absolute value of the Love number is frequency dependent.

## 5.2 Tidal damping in the Moon, from laser ranging

Fitting of the LLR data to the power scaling law (79), which was carried out by Williams et al. (2001), has demonstrated that the lunar mantle possesses quite an abnormal value of the exponent:  $-0.19$ . A later reexamination in Williams et al. (2008) rendered a less embarrassing value,  $-0.09$ , which nevertheless was still negative and thus seemed to contradict our knowledge about microphysical damping mechanisms in minerals. Thereupon, Williams & Boggs (2009) commented:

*“There is a weak dependence of tidal specific dissipation  $Q$  on period. The  $Q$  increases from  $\sim 30$  at a month to  $\sim 35$  at one year.  $Q$  for rock is expected to have a weak dependence on tidal period, but it is expected to decrease with period rather than increase. The frequency dependence of  $Q$  deserves further attention and should be improved.”*

While there always remains a possibility of the raw data being insufficient or of the fitting procedure being imperfect, the fact is that the negative exponent obtained in *Ibid.* does **not** necessarily contradict the scaling law (79) proven for minerals and partial melts. Indeed, the exponent obtained by the LLR Team was not the  $\alpha$  from (79) but was the  $p$  from (80). The distinction is critical due to the difference in frequency-dependence of the seismic and tidal dissipation. It turns out that the near-viscous value  $p \sim 1$  from the second line of (80), appropriate for low frequencies, does not retain its value all the way to the zero frequency. Specifically, in subsection 5.4 we shall see that at the frequency  $\frac{1}{\tau_M A_l}$  (where  $\tau_M = \eta/\mu$  is the Maxwell time, with  $\eta$  and  $\mu$  being the lunar mantle’s viscosity and rigidity), the exponent  $p$  begins

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<sup>9</sup> This circumstance was ignored by Defraigne & Smits (1999). Accordingly, if the claims by Karato & Spetzler (1990) are correct, the table of corrections for the tidal variations in the Earth’s rotation in the IERS Conventions is likely to contain increasing errors for periods of about a year and longer.

This detail is missing in the theory of the Chandler wobble of Mars, by Zharkov & Gudkova (2009).

to decrease with the decrease of the frequency. As the frequency becomes lower,  $p$  changes its sign and eventually becomes  $-1$  in a close vicinity of  $\chi = 0$ . This behaviour follows from calculations based on a realistic rheology (see formulae (92 - 94) below), and it goes along well with the evident physical fact that the average tidal torque must vanish in a resonance.<sup>10</sup> In subsection 5.7, comparison of this behaviour with the LLR results will yield us an estimate for the mean lunar viscosity.

### 5.3 The Andrade model as an example of viscoelastic behaviour

The complex compliance of a Maxwell material contains a term  $J = J(0)$  responsible for the elastic part of the deformation and a term  $-\frac{i}{\chi\eta}$  describing the viscosity. Whatever other terms get incorporated into the compliance, these will correspond to other forms of hereditary reaction. The available geophysical data strongly favour a particular extension of the Maxwell approach, the *Andrade model* (Cottrell & Aytakin 1947, Duval 1976). In modern notations, the model can be expressed as<sup>11</sup>

$$J(t-t') = [J + \beta(t-t')^\alpha + \eta^{-1}(t-t')] \Theta(t-t') \quad , \quad (81)$$

$\alpha$  being a dimensionless parameter,  $\beta$  being a dimensional parameter,  $\eta$  denoting the steady-state viscosity, and  $J$  standing for the unrelaxed compliance, which is inverse to the unrelaxed rigidity:  $J \equiv J(0) = 1/\mu(0) = 1/\mu$ . We see that (81) is the Maxwell model amended with an extra term of a hereditary nature.

A simple example illustrating how the model works is rendered by deformation under constant loading. In this case, the anelastic term dominates at short times, the strain thus being a convex function of  $t$  (the so-called primary or transient creep). As time goes on and the applied loading is kept constant, the viscous term becomes larger, and the strain becomes almost linear in time – a phenomenon called the secondary creep.

Remarkably, for all minerals (including ices) the values of  $\alpha$  belong to the interval from 0.14 through 0.4 (more often, through 0.3) – see the references in subsection 5.1 above. The other parameter,  $\beta$ , may be rewritten as

$$\beta = J \tau_A^{-\alpha} = \mu^{-1} \tau_A^{-\alpha} \quad , \quad (82)$$

the quantity  $\tau_A$  having dimensions of time. This quantity is the timescale associated with the Andrade creep, and it may be termed as the “Andrade time” or the “anelastic time”. It is clear from (82) that a short  $\tau_A$  makes the anelasticity more pronounced, while a long  $\tau_A$  makes the anelasticity weak.<sup>12</sup>

It is known from Castillo-Rogez et al. (2011) and Castillo-Rogez & Choukroun (2010) that for some minerals, within some frequency bands, the Andrade time gets very close to the Maxwell time:

$$\tau_A \approx \tau_M \quad \implies \quad \beta \approx J \tau_M^{-\alpha} = J^{1-\alpha} \eta^{-\alpha} = \mu^{\alpha-1} \eta^{-\alpha} \quad , \quad (83)$$

<sup>10</sup> For example, the principal tidal torque  $\tau_{lmpq} = \tau_{2200}$  acting on a secondary must vanish when the secondary is crossing the synchronous orbit. Naturally, this happens because  $p$  becomes  $-1$  in the close vicinity of  $\chi_{2200} = 0$ .

<sup>11</sup> As long as we agree to integrate over  $t-t' \in [0, \infty)$ , the terms  $\beta(t-t')^\alpha$  and  $\eta^{-1}(t-t')$  can do without the Heaviside step-function  $\Theta(t-t')$ . We remind though that the first term,  $J$ , does need this multiplier, so that insertion of (81) into (43) renders the desired  $J\delta(t-t')$  under the integral, after the differentiation in (43) is performed.

<sup>12</sup> While the Andrade creep is likely to be caused by “unpinning” of jammed dislocations (Karato & Spetzler 1990, Miguel et al 2002), it is not apparently clear if the Andrade time can be identified with the typical time of unpinning of defects.

where the relaxation Maxwell time is given by:

$$\tau_M \equiv \frac{\eta}{\mu} = \eta J \quad . \quad (84)$$

On general grounds, though, one cannot expect the anelastic timescale  $\tau_A$  and the viscoelastic timescale  $\tau_M$  to coincide in all situations. This is especially so due to the fact that both these times may possess some degree of frequency-dependence. Specifically, there exist indications that in the Earth's mantle the role of anelasticity (compared to viscoelasticity) undergoes a decrease when the frequencies become lower than 1/yr – see the microphysical model suggested in subsection 5.2.3 of Karato & Spetzler (1990). It should be remembered, though, that the relation between  $\tau_A$  and  $\tau_M$  may depend also upon the intensity of loading, i.e., upon the damping mechanisms involved. The microphysical model considered in *Ibid.* was applicable to strong deformations, with anelastic dissipation being dominated by dislocations unpinning. Accordingly, the dominance of viscosity over anelasticity ( $\tau_A \ll \tau_M$ ) at low frequencies may be regarded proven for strong deformations only. At low stresses, when the grain-boundary diffusion mechanism is dominant, the values of  $\tau_A$  and  $\tau_M$  may remain comparable at low frequencies. The topic needs further research.

In terms of the Andrade and Maxwell times, the compliance becomes:

$$J(t - t') = J \left[ 1 + \left( \frac{t - t'}{\tau_A} \right)^\alpha + \frac{t - t'}{\tau_M} \right] \Theta(t - t') \quad . \quad (85)$$

In the frequency domain, compliance (85) will look:

$$\bar{J}(\chi) = J + \beta (i\chi)^{-\alpha} \Gamma(1 + \alpha) - \frac{i}{\eta\chi} \quad (86a)$$

$$= J \left[ 1 + (i\chi\tau_A)^{-\alpha} \Gamma(1 + \alpha) - i(\chi\tau_M)^{-1} \right] \quad , \quad (86b)$$

$\chi$  being the frequency, and  $\Gamma$  denoting the Gamma function. The imaginary and real parts of the complex compliance are:

$$\mathcal{Im}[\bar{J}(\chi)] = -\frac{1}{\eta\chi} - \chi^{-\alpha} \beta \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1) \quad (87a)$$

$$= -J(\chi\tau_M)^{-1} - J(\chi\tau_A)^{-\alpha} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1) \quad (87b)$$

and

$$\mathcal{Re}[\bar{J}(\chi)] = J + \chi^{-\alpha} \beta \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1) \quad (88a)$$

$$= J + J(\chi\tau_A)^{-\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1) \quad , \quad (88b)$$

whence we obtain the following dependence of the phase lag upon the frequency:

$$\tan \delta(\chi) = -\frac{\mathcal{Im}[\bar{J}(\chi)]}{\mathcal{Re}[\bar{J}(\chi)]} = \frac{(\eta\chi)^{-1} + \chi^{-\alpha} \beta \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1)}{\mu^{-1} + \chi^{-\alpha} \beta \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1)} \quad (89a)$$

$$= \frac{z^{-1} \zeta + z^{-\alpha} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1)}{1 + z^{-\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha + 1)} \quad . \quad (89b)$$

Here  $z$  is the dimensionless frequency defined as

$$z \equiv \chi \tau_A = \chi \tau_M \zeta \quad , \quad (90)$$

while  $\zeta$  is a dimensionless parameter of the Andrade model:

$$\zeta \equiv \frac{\tau_A}{\tau_M} \quad . \quad (91)$$

## 5.4 Tidal response of viscoelastic near-spherical bodies obeying the Andrade and Maxwell models

An  $lmpq$  term in the expansion for the tidal torque is proportional to the factor  $k_l(\chi) \sin \epsilon_l(\chi) = |\bar{k}_l(\chi_{lmpq})| \sin \epsilon_l(\chi_{lmpq})$ . Hence the tidal response of a body is determined by the frequency-dependence of these factors.

Combining (72) with (86), and keeping in mind that  $A_l \gg 1$ , it is easy to write down the frequency-dependencies of the products  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$ . Referring the reader to Appendix E.2 for details, we present the results, without the sign multiplier.

- In the high-frequency band:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) \approx \frac{3}{2(l-1)} \frac{A_l}{(A_l+1)^2} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha+1) \zeta^{-\alpha} (\tau_M \chi)^{-\alpha} \quad , \quad \text{for} \quad \chi \gg \tau_M^{-1} \quad . \quad (92)$$

For small bodies and small terrestrial planets (i.e., for  $A_l \gg 1$ ), the boundary between the high and intermediate frequencies turns out to be

$$\chi_{HI} = \tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}} \quad .$$

For large terrestrial planets (i.e., for  $A_l \ll 1$ ) the boundary frequency is

$$\chi_{HI} = \tau_A^{-1} = \tau_M^{-1} \zeta^{-1} \quad .$$

At high frequencies, anelasticity dominates. So, dependent upon the microphysics of the mantle, the parameter  $\zeta$  may be of order unity or slightly lower. We say *slightly*, because we expect both anelasticity and viscosity to be present near the transitional zone. (A too low  $\zeta$  would eliminate viscosity from the picture completely.) This said, we may assume that the boundary  $\chi_{HI}$  is comparable to  $\tau_M^{-1}$  for both small and large solid objects. This is why in (92) we set the inequality simply as  $\chi \gg \tau_M^{-1}$ .

- In the intermediate-frequency band:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) \approx \frac{3}{2(l-1)} \frac{A_l}{(A_l+1)^2} (\tau_M \chi)^{-1} \quad , \quad \text{for} \quad \tau_M^{-1} \gg \chi \gg \tau_M^{-1} (A_l+1)^{-1} \quad . \quad (93)$$

While the consideration in the Appendix E.2 renders  $\tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}}$  for the upper bound, here we approximate it with  $\tau_M^{-1}$  in understanding that  $\zeta$  does not differ from unity too much near the transitional zone. Further advances of rheology may challenge this convenient simplification.

- In the low-frequency band:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) \approx \frac{3}{2(l-1)} A_l \tau_M \chi \quad , \quad \text{for} \quad \tau_M^{-1} (A_l + 1)^{-1} \gg \chi \quad . \quad (94)$$

Scaling laws (92) and (93) mimic, up to constant factors, the frequency-dependencies of  $|\bar{J}(\chi)| \sin \delta(\chi) = -\mathcal{I}m[\bar{J}(\chi)]$  at high and low frequencies, correspondingly, – this can be seen from (87).

Expression (94) however shows a remarkable phenomenon inherent only in the *tidal* lagging, and not in the lagging in a sample of material: at frequencies below  $\tau_M^{-1} (A_l + 1)^{-1} = \frac{\mu}{\eta} (A_l + 1)^{-1}$ , the product  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$  changes its behaviour and becomes linear in  $\chi$ .

While elsewhere the  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$  factor increases with decreasing  $\chi$ , it changes its behaviour drastically on close approach to the zero frequency. Having reached a finite maximum at about  $\chi = \tau_M^{-1} (A_l + 1)^{-1}$ , the said factor begins to scale linearly in  $\chi$  as  $\chi$  approaches zero. This way, the factor  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$  decreases continuously on close approach to a resonance, becomes nil together with the frequency at the point of resonance. So neither the tidal torque nor the tidal force explodes in resonances. In a somewhat heuristic manner, this change in the frequency-dependence was pointed out, for  $l = 2$ , in Section 9 of Efroimsky & Williams (2009).

## 5.5 Example

Figure 1 shows the absolute value,  $k_2 \equiv |\bar{k}_2(\chi)|$ , as well as the real part,  $\mathcal{R}e[\bar{k}_2(\chi)] = k_2 \cos \epsilon_2$ , and the negative imaginary part,  $-\mathcal{I}m[\bar{k}_2(\chi)] = k_2 \sin \epsilon_2$ , of the complex quadrupole Love number. Each of the three quantities is represented by its decadic logarithm as a function of the decadic logarithm of the forcing frequency  $\chi$  (given in Hz). The curves were obtained by insertion of formulae (87 - 88) into (68). As an example, the case of  $-\mathcal{I}m[\bar{k}_2(\chi)]$  is worked out in Appendix E.2, see formulae (252 - 88).

Both in the high- and low-frequency limits, the negative imaginary part of  $\bar{k}_2(\chi)$ , given on Figure 1 by the red curve, approaches zero. Accordingly, over the low- and high-frequency bands the real part (the green line) virtually coincides with the absolute value (the blue line).

While on the left and on the close right of the peak, dissipation is mainly due to viscosity, friction at higher frequencies is mainly due to anelasticity. This switch corresponds to the change of the slope of the red curve at high frequencies (for our choice of parameters, at around  $10^{-5}$  Hz). This change of the slope is often called *the elbow*.

Figure 1 was generated for  $A_2 = 80.5$  and  $\tau_M = 3.75 \times 10^5$  s. The value of  $A_2$  corresponds to the Moon modeled by a homogeneous sphere of rigidity  $\mu = 0.8 \times 10^{11}$  Pa. Our choice of the value of  $\tau_M \equiv \eta/\mu$  corresponds to a homogeneous Moon with the said value of rigidity and with viscosity set to be  $\eta = 3 \times 10^{16}$  Pa s. The reason why we consider an example with such a low value of  $\eta$  will be explained in subsection 5.7. Finally, it was assumed for simplicity that  $\zeta = 1$ , i.e., that  $\tau_A = \tau_M$ . Although unphysical at low frequencies, this simplification only slightly changes the shape of the “elbow” and exerts virtually no influence upon the maximum of the red curve, provided the maximum is located well into the viscosity zone.

## 5.6 Crossing a resonance – with a chance for entrapment

As ever, we recall that in the expansion for the tidal torque factors (92 - 94) should appear in the company of multipliers  $\text{sgn } \omega$ . For example, the factor (94) describing dissipation near an *lmpq* resonance will enter the expansions as

$$|\bar{k}_l(\chi_{lmpq})| \sin \epsilon_l(\chi_{lmpq}) \text{sgn } \omega_{lmpq} \approx \frac{3}{2(l-1)} A_l \tau_M \chi_{lmpq} \text{sgn } \omega_{lmpq} = \frac{3}{2(l-1)} A_l \tau_M \omega_{lmpq} \quad . \quad (95)$$



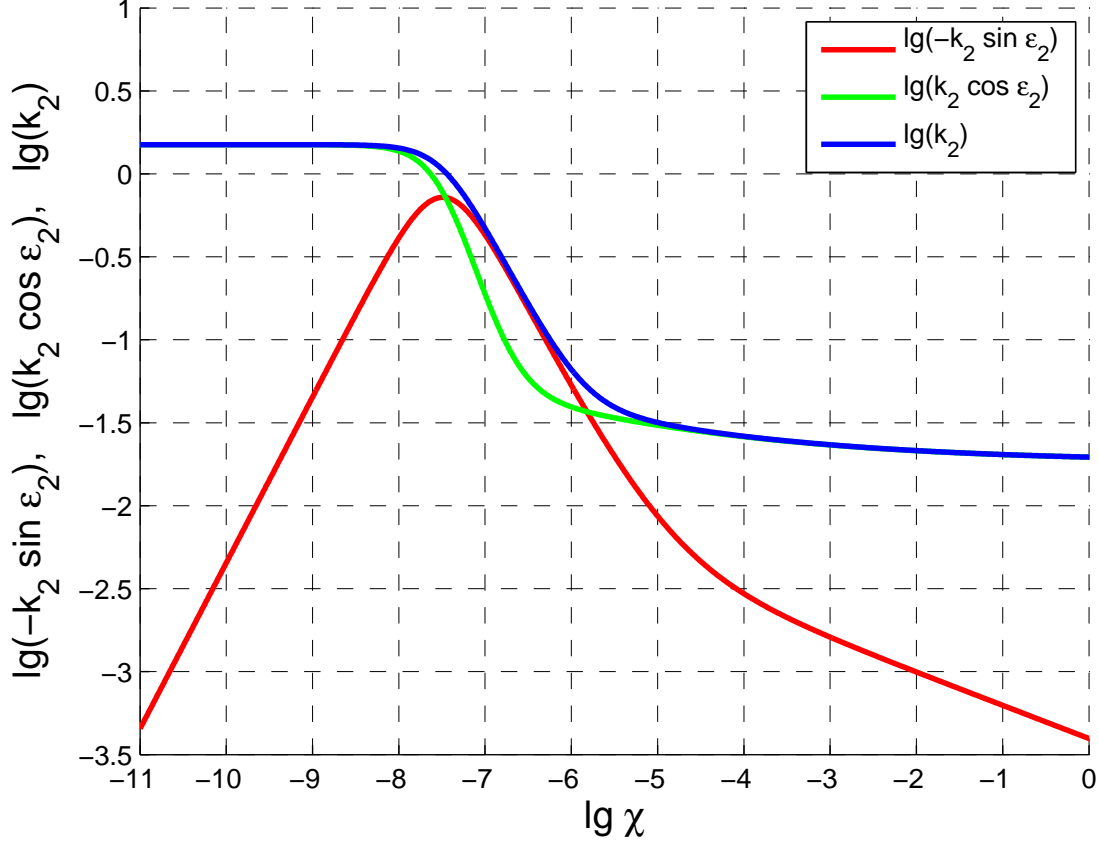


Figure 1: Tidal response of a homogeneous spherical Andrade body, set against the decadic logarithm of the forcing frequency  $\chi$  (in Hz). The blue curve renders the decadic logarithm of the absolute value of the quadrupole complex Love number,  $\lg k_2 = \lg |\bar{k}_2(\chi)|$ . The green and red curves depict the logarithms of the real and the negative imaginary parts of the Love number:  $\lg \mathcal{Re} [\bar{k}_2(\chi)] = \lg (k_2 \cos \epsilon_2)$  and  $\lg \{-\mathcal{Im} [\bar{k}_2(\chi)]\} = \lg (-k_2 \sin \epsilon_2)$ , accordingly. The change in the slope of the red curve (the “elbow”), which takes place to the right of the maximum, corresponds to the switch from viscosity dominance at lower frequencies to anelasticity dominance at higher frequencies. The parameters  $A_2$  and  $\tau_M$  were given values appropriate to a homogeneous Moon with a low viscosity, as described in subsection 5.7. The plots were generated for an Andrade body with  $\zeta = 1$  at all frequencies. Setting the body Maxwell at lower frequencies will only slightly change the shape of the “elbow” and will have virtually no effect on the maximum.

Naturally, the  $lmpq$  term of the torque depends on  $\omega_{lmpq}$  and not just on  $\chi_{lmpq} = |\omega_{lmpq}|$ . This term then goes continuously through zero, and changes its sign as the  $lmpq$  resonance is crossed (i.e., as  $\omega_{lmpq}$  goes through nil and changes its sign).

Formula (95) tells us that an  $lmpq$  component of the tidal torque goes continuously through zero as the satellite is traversing the commensurability which corresponds to vanishing of a tidal frequency  $\chi_{lmpq}$ . This gets along with the physically evident fact that the principal (i.e., 2200) term of the tidal torque should vanish as the secondary approaches the synchronous orbit.

It is important that a  $lmpq$  term of the torque changes its sign and thus creates a chance for entrapment. As the value of an  $lmpq$  term of the torque is much lower than that of the principal, 2200 term, we see that a perfectly spherical body will never get stuck in a resonance other than 2200. (The latter is, of course, the 1 : 1 resonance, i.e., the one in which the principal term of the torque vanishes.) However, the presence of the triaxiality-generated torque is known to contribute to the probabilities of entrapment into other resonances (provided the eccentricity is not zero). Typically, in the literature they consider a superposition of the triaxiality-generated torque with the principal tidal term. We would point out that the “trap” shape of the  $lmpq$  term (95) makes this term relevant for the study of entrapment in the  $lmpq$  resonance. In some situations, one has to take into account also the non-principal terms of the tidal torque.

## 5.7 Comparison with the LLR results

As we mentioned above, fitting of the lunar laser ranging (LLR) data to the power law has resulted in a very small *negative* exponent  $p = -0.19$  (Williams et al. 2001). Since the measurements of the lunar damping described in *Ibid.* rendered information on the *tidal* and not seismic dissipation, those results can and should be compared to the scaling law (92 - 94). As the small negative exponent was devised from observations over periods of a month to a year, it is natural to presume that the appropriate frequencies were close to or slightly below the frequency  $\frac{1}{\tau_M A_2}$  at which the factor  $k_2 \sin \epsilon_2$  has its peak:

$$3 \times 10^6 \text{ s} \approx 0.1 \text{ yr} = \tau_M A_2 = \frac{\eta}{\mu} A_2 = \frac{57 \eta}{8 \pi G (\rho R)^2} \quad , \quad (96)$$

as on Figure 1. Hence, were the Moon a uniform viscoelastic body, its viscosity would be only

$$\eta = 3 \times 10^{16} \text{ Pa s} \quad . \quad (97)$$

For the actual Moon, the estimate means that the lower mantle contains a high percentage of partial melt, a fact which goes along well with the model suggested in Weber et al. (2011), and which was anticipated yet in Williams et al. (2001) and Williams & Boggs (2009), following an earlier work by Nakamura et al. (1974).

## 6 The polar component of the tidal torque acting on the primary

Let vector  $\vec{r} = (r, \lambda, \phi)$  point from the centre of the primary toward a point-like secondary of mass  $M_{sec}$ . Associating the coordinate system with the primary, we reckon the latitude  $\phi$  from the equator. Setting the coordinate system to corotate, we determine the longitude  $\lambda$  from a fixed meridian. The tidally induced component of the primary’s potential,  $U$ , can be generated either by this secondary itself or by some other secondary of mass  $M_{sec}^*$  located at  $\vec{r}^* = (r^*, \lambda^*, \phi^*)$ . In either situation, the tidally induced potential  $U$  generates a tidal force and a tidal torque wherewith the secondary of mass  $M_{sec}$  acts on the primary.

The scope of this paper is limited to low values of  $i$ . When the role of the primary is played by a planet, the secondary being its satellite,  $i$  is the satellite's inclination. When the role of the primary is played by the satellite, the planet acting as its secondary,  $i$  acquires the meaning of the satellite's obliquity. Similarly, when the planet is regarded as a primary and its host star is treated as its secondary,  $i$  is the obliquity of the planet. In all these cases, the smallness of  $i$  indicates that the tidal torque acting on the primary can be identified with its polar component, the one orthogonal to the equator of the primary. The other components of the torque will be neglected in this approximation.

The polar component of the torque acting on the primary is the negative of the partial derivative of the tidal potential, with respect to the primary's sidereal angle:

$$\mathcal{T}(\vec{r}) = - M_{sec} \frac{\partial U(\vec{r})}{\partial \theta} , \quad (98)$$

$\theta$  standing for the primary's sidereal angle. This formula is convenient when the tidal potential  $U$  is expressed through the secondary's orbital elements and the primary's sidereal angle.<sup>13</sup>

Here and hereafter we are deliberately referring to *a primary and a secondary* in lieu of *a planet and a satellite*. The preference stems from our intention to extend the formalism to setting where a moon is playing the role of a tidally-perturbed primary, the planet being its tide-producing secondary. Similarly, when addressing the rotation of Mercury, we interpret the Sun as a secondary that is causing a tide on the effectively primary body, Mercury.

## 7 The tidal potential

### 7.1 Darwin (1879) and Kaula (1964)

The potential produced at point  $\vec{R} = (R, \lambda, \phi)$  by a secondary body of mass  $M^*$ , located at  $\vec{r}^* = (r^*, \lambda^*, \phi^*)$  with  $r^* \geq R$ , is given by (1). When a tide-raising secondary located at  $\vec{r}^*$  distorts the shape of the primary, the potential generated by the primary at some exterior point  $\vec{r}$  gets changed. In the linear approximation, its variation is given by (2). Insertion of (1) into (2) entails

$$U(\vec{r}) = -G M_{sec}^* \sum_{l=2}^{\infty} k_l \frac{R^{2l+1}}{r^{l+1} r^{*l+1}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi) P_{lm}(\sin \phi^*) \cos m(\lambda - \lambda^*) . \quad (99)$$

A different expression for the tidal potential was offered by Kaula (1961, 1964), who developed a powerful technique that enabled him to switch from the spherical coordinates to the Kepler elements  $(a^*, e^*, i^*, \Omega^*, \omega^*, \mathcal{M}^*)$  and  $(a, e, i, \Omega, \omega, \mathcal{M})$  of the secondaries located at  $\vec{r}^*$  and

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<sup>13</sup> Were the potential written down in the spherical coordinates associated with the primary's equator and corotating with the primary, the polar component of the tidal torque could be calculated with aid of the expression

$$\mathcal{T}(\vec{r}) = M_{sec} \frac{\partial U(\vec{r})}{\partial \lambda}$$

derived, for example, in Williams & Efroimsky (2012). That the expression agrees with (98) can be seen from the formula

$$\lambda = -\theta + \Omega + \omega + \nu + O(i^2) = -\theta + \Omega + \omega + \mathcal{M} + 2e \sin \mathcal{M} + O(e^2) + O(i^2) ,$$

$e, i, \omega, \Omega, \nu$  and  $\mathcal{M}$  being the eccentricity, inclination, argument of the pericentre, longitude of the node, true anomaly, and mean anomaly of the tide-raising secondary.

$\vec{r}$ . Application of this technique to (99) results in

$$U(\vec{r}) = - \sum_{l=2}^{\infty} k_l \left( \frac{R}{a} \right)^{l+1} \frac{G M_{sec}^*}{a^*} \left( \frac{R}{a^*} \right)^l \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) \sum_{p=0}^l F_{lmp}(i^*) \sum_{q=-\infty}^{\infty} G_{lpq}(e^*) \sum_{h=0}^l F_{lmh}(i) \sum_{j=-\infty}^{\infty} G_{lhj}(e) \cos [(v_{lmpq}^* - m\theta^*) - (v_{lmhj} - m\theta)] , \quad (100)$$

where

$$v_{lmpq}^* \equiv (l-2p)\omega^* + (l-2p+q)\mathcal{M}^* + m\Omega^* , \quad (101)$$

$$v_{lmhj} \equiv (l-2h)\omega + (l-2h+j)\mathcal{M} + m\Omega , \quad (102)$$

$\theta = \theta^*$  being the sidereal angle,  $G_{lpq}(e)$  signifying the eccentricity functions,<sup>14</sup> and  $F_{lmp}(i)$  denoting the inclination functions (Gooding & Wagner 2008).

Being equivalent for a planet with an instant response of the shape, (99) and (100) disagree when dissipation-caused delays come into play. Kaula's expression (100), as well as its truncated, Darwin's version,<sup>15</sup> is capable of accommodating separate phase lags for each mode:

$$U(\vec{r}) = - \sum_{l=2}^{\infty} k_l \left( \frac{R}{a} \right)^{l+1} \frac{G M_{sec}^*}{a^*} \left( \frac{R}{a^*} \right)^l \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) \sum_{p=0}^l F_{lmp}(i^*) \sum_{q=-\infty}^{\infty} G_{lpq}(e^*) \sum_{h=0}^l F_{lmh}(i) \sum_{j=-\infty}^{\infty} G_{lhj}(e) \cos [(v_{lmpq}^* - m\theta^*) - (v_{lmhj} - m\theta) - \epsilon_{lmpq}] , \quad (103)$$

where

$$\epsilon_{lmpq} = \left[ (l-2p)\dot{\omega}^* + (l-2p+q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*) \right] \Delta t_{lmpq} = \omega_{lmpq}^* \Delta t_{lmpq} = \pm \chi_{lmpq}^* \Delta t_{lmpq} \quad (104)$$

is the phase lag. The tidal mode  $\omega_{lmpq}^*$  introduced in (104) is

$$\omega_{lmpq}^* \equiv (l-2p)\dot{\omega}^* + (l-2p+q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*) , \quad (105)$$

while the positively-defined quantity

$$\chi_{lmpq}^* \equiv |\omega_{lmpq}^*| = |(l-2p)\dot{\omega}^* + (l-2p+q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*)| \quad (106)$$

is the actual physical  $lmpq$  frequency excited by the tide in the primary. The corresponding positively-defined time delay  $\Delta t_{lmpq} = \Delta t_l(\chi_{lmpq}^*)$  depends on this physical frequency, the functional forms of this dependence being different for different materials.

<sup>14</sup> Functions  $G_{lhj}(e)$  coincide with the Hansen polynomials  $X_{(l-2p+q)}^{(-l-1), (l-2p)}(e)$ . In Appendix G, we provide a table of the  $G_{lhj}(e)$  required for expansion of tides up to  $e^6$ , inclusively.

<sup>15</sup> While the treatment by Kaula (1964) entails the infinite Fourier series (100), the development by Darwin (1879, 1880) renders its partial sum with  $|l|, |q|, |j| \leq 2$ . For a simple introduction into Darwin's method see Ferraz-Mello et al. (2008). Be mindful that in *Ibid.* the convention on the notations  $\vec{r}$  and  $\vec{r}^*$  is opposite to ours.

In neglect of the apsidal and nodal precessions, and also of  $\dot{\mathcal{M}}_0$ , the above formulae become:

$$\omega_{lmpq} = (l - 2p + q) n - m \dot{\theta} \quad , \quad (107)$$

$$\chi_{lmpq} \equiv |\omega_{lmpq}| = |(l - 2p + q) n - m \dot{\theta}| \quad , \quad (108)$$

and

$$\epsilon_{lmpq} \equiv \omega_{lmpq} \Delta t_{lmpq} = \left[ (l - 2p + q) n - m \dot{\theta} \right] \Delta t_{lmpq} \quad (109a)$$

$$= \chi_{lmpq} \Delta t_l(\chi_{lmpq}) \operatorname{sgn} \left[ (l - 2p + q) n - m \dot{\theta} \right] \quad , \quad (109b)$$

Formulae (100) and (103) constitute the pivotal result of Kaula's theory of tides. Importantly, Kaula's theory imposes no *a priori* constraint on the form of frequency-dependence of the lags.

## 8 The Darwin torque

As explained in Williams & Efroimsky (2012), the empirical model by MacDonald (1964), called *MacDonald torque*, tacitly sets an unphysical rheology of the satellite's material. The rheology is given by (79) with  $\alpha = -1$ . More realistic is the dissipation law (80). An even more accurate and practical formulation of the damping law, stemming from the Andrade formula for the compliance, is rendered by (92 - 94). These formulae should be inserted into the Darwin-Kaula theory of tides.

### 8.1 The secular and oscillating parts of the Darwin torque

#### 8.1.1 The general formula

Direct differentiation of (103) with respect to  $-\theta$  will result in the expression<sup>16</sup>

$$\begin{aligned} \mathcal{T} = & - \sum_{l=2}^{\infty} \left( \frac{R}{a} \right)^{l+1} \frac{G M_{sec}^* M_{sec}}{a^*} \left( \frac{R}{a^*} \right)^l \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} 2m \sum_{p=0}^l F_{lmp}(i^*) \sum_{q=-\infty}^{\infty} G_{lpq}(e^*) \\ & \sum_{h=0}^l F_{lmh}(i) \sum_{j=-\infty}^{\infty} G_{lhj}(e) k_l \sin \left[ v_{lmpq}^* - v_{lmhj} - \epsilon_{lmpq} \right] \quad . \end{aligned} \quad (110)$$

If the tidally-perturbed and tide-raising secondaries are the same body, then  $M_{sec} = M_{sec}^*$ , and all the elements coincide with their counterparts with an asterisk. Hence the differences

$$v_{lmpq}^* - v_{lmhj} =$$

$$(l - 2p + q) \mathcal{M}^* - (l - 2h + j) \mathcal{M} + m(\Omega^* - \Omega) + l(\omega^* - \omega) - 2p\omega^* + 2h\omega \quad (111)$$

get simplified to

$$v_{lmpq}^* - v_{lmhj} = (2h - 2p + q - j) \mathcal{M}^* + (2h - 2p) \omega^* \quad , \quad (112)$$

an expression containing both short-period contributions proportional to the mean anomaly, and long-period contributions proportional to the argument of the pericentre.

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<sup>16</sup> For justification of this operation, see Section 6 in Efroimsky & Williams (2009).

### 8.1.2 The secular, the purely-short-period, and the mixed-period parts of the torque

Now we see that the terms entering series (110) can be split into three groups:

(1) The terms, in which indices  $(p, q)$  coincide with  $(h, j)$ , constitute a secular part of the tidal torque, because in such terms  $v_{lmhj}$  cancel with  $v_{lmpq}^*$ . This  $\mathcal{M}$ - and  $\omega$ -independent part is furnished by

$$\bar{\mathcal{T}} = \sum_{l=2}^{\infty} 2 G M_{sec}^2 \frac{R^{2l+1}}{a^{2l+2}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} m \sum_{p=0}^l F_{lmp}^2(i) \sum_{q=-\infty}^{\infty} G_{lpq}^2(e) k_l \sin \epsilon_{lmpq} . \quad (113)$$

(2) The terms with  $p = h$  and  $q \neq j$  constitute a purely short-period part of the torque:

$$\begin{aligned} \tilde{\mathcal{T}} = - \sum_{l=2}^{\infty} 2 G M_{sec}^2 \frac{R^{2l+1}}{a^{2l+2}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} m \sum_{p=0}^l F_{lmp}^2(i) \sum_{q=-\infty}^{\infty} \sum_{\substack{j=-\infty \\ j \neq q}}^{\infty} G_{lpq}(e) G_{lpj}(e) k_l \sin [(q \\ - j) \mathcal{M} - \epsilon_{lmpq}] . \end{aligned} \quad (114)$$

(3) The remaining terms, ones with  $p \neq h$ , make a mixed-period part comprised of both short- and long-period terms:

$$\begin{aligned} \mathcal{T}^{mixed} = & - \sum_{l=2}^{\infty} 2 G M_{sec}^2 \frac{R^{2l+1}}{a^{2l+2}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} m \sum_{p=0}^l F_{lmp}(i) \sum_{\substack{h=0 \\ h \neq p}}^l F_{lmh}(i) \sum_{q=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} G_{lhq}(e) G_{lpj}(e) k_l \sin [(2h \\ - 2p + q - j) \mathcal{M}^* + (2h - 2p) \omega^* - \epsilon_{lmpq}] . \end{aligned} \quad (115)$$

### 8.1.3 The $l = 2$ and $l = 3$ terms in the $O(i^2)$ approximation

For  $l = 2$ , index  $m$  will take the values 0, 1, 2 only. Although the  $m = 0$  terms enter the potential, they add nothing to the torque, because differentiation of (103) with respect to  $-\theta$  furnishes the  $m$  multiplier in (110). To examine the remaining terms, we should consider the inclination functions with subscripts  $(lmp) = (220), (210), (211)$  only:

$$F_{220}(i) = 3 + O(i^2) , \quad F_{210}(i) = \frac{3}{2} \sin i + O(i^2) , \quad F_{211}(i) = -\frac{3}{2} \sin i + O(i^2) , \quad (116)$$

all the other  $F_{2mp}(i)$  being of order  $O(i^2)$  or higher. Thence for  $p = h$  (i.e., both in the secular and purely-short-period parts) it is sufficient, in the  $O(i^2)$  approximation, to keep only the terms with  $F_{220}^2(i)$ , ignoring those with  $F_{210}^2(i)$  and  $F_{211}^2(i)$ . We see that in the  $O(i^2)$  approximation

- among the  $l = 2$  terms, both in the secular and purely short-period parts, only the terms with  $(lmp) = (220)$  are relevant.

In the case of  $p \neq h$ , i.e., in the mixed-period part, the terms of the leading order in inclination are:  $F_{lmp}(i) F_{lmh}(i) = F_{210}(i) F_{211}(i)$  and  $F_{lmp}(i) F_{lmh}(i) = F_{211}(i) F_{210}(i)$ , which happen to be equal to one another, and to be of order  $O(i^2)$ . This way, in the  $O(i^2)$  approximation,

- the mixed-period part of the  $l = 2$  component may be omitted.

The inclination functions  $F_{lmp} = F_{310}, F_{312}, F_{313}, F_{320}, F_{321}, F_{322}, F_{323}, F_{331}, F_{332}, F_{333}$  are of order  $O(i)$  or higher. The terms containing the squares or cross-products of these functions may thus be dropped. Specifically, the smallness of the cross-terms tells us that

- the mixed-period part of the  $l = 3$  component may be omitted.

What remains is the terms containing the squares of functions

$$F_{311}(i) = -\frac{3}{2} + O(i^2) \quad \text{and} \quad F_{330}(i) = 15 + O(i^2) \quad . \quad (117)$$

In other words,

- among the  $l = 3$  terms, both in the secular and purely short-period parts, only the terms with  $(lmp) = (311)$  and  $(lmp) = (330)$  are important.

All in all, for  $l = 2$  and  $l = 3$  the mixed-period parts of the torque may be neglected, in the  $O(i^2)$  approximation. The surviving terms of the secular and the purely short-period parts will be developed up to  $e^6$ , inclusively.

## 8.2 Approximation for the secular and short-period parts of the tidal torque

As we just saw, both the secular and short-period parts of the torque may be approximated with the following degree-2 and degree-3 components:

$$\begin{aligned} \overline{\mathcal{T}} &= \overline{\mathcal{T}}_{l=2} + \overline{\mathcal{T}}_{l=3} + O(\epsilon(R/a)^9) \\ &= \overline{\mathcal{T}}_{(lmp)=(220)} + \left[ \overline{\mathcal{T}}_{(lmp)=(311)} + \overline{\mathcal{T}}_{(lmp)=(330)} \right] + O(\epsilon i^2) + O(\epsilon(R/a)^9) \quad , \end{aligned} \quad (118)$$

and

$$\begin{aligned} \widetilde{\mathcal{T}} &= \widetilde{\mathcal{T}}_{l=2} + \widetilde{\mathcal{T}}_{l=3} + O(\epsilon(R/a)^9) \\ &= \widetilde{\mathcal{T}}_{(lmp)=(220)} + \left[ \widetilde{\mathcal{T}}_{(lmp)=(311)} + \widetilde{\mathcal{T}}_{(lmp)=(330)} \right] + O(\epsilon i^2) + O(\epsilon(R/a)^9) \quad , \end{aligned} \quad (119)$$

were the  $l = 2$  and  $l = 3$  inputs are of the order  $(R/a)^5$  and  $(R/a)^7$ , accordingly; while the  $l = 4, 5, \dots$  inputs constitute  $O(\epsilon(R/a)^9)$ .

Expressions for  $\overline{\mathcal{T}}_{(lmp)=(220)}$ ,  $\overline{\mathcal{T}}_{(lmp)=(311)}$ , and  $\overline{\mathcal{T}}_{(lmp)=(330)}$  are furnished by formulae (283),

(285), and (287) in Appendix H. As an example, here we provide one of these components:

$$\begin{aligned}
\overline{\mathcal{T}}_{(lmp)=(220)} &= \frac{3}{2} G M_{sec}^2 R^5 a^{-6} \left[ \frac{1}{2304} e^6 k_2 \sin |\epsilon_{220-3}| \operatorname{sgn} \left( -n - 2\dot{\theta} \right) \right. \\
&+ \left( \frac{1}{4} e^2 - \frac{1}{16} e^4 + \frac{13}{768} e^6 \right) k_2 \sin |\epsilon_{220-1}| \operatorname{sgn} \left( n - 2\dot{\theta} \right) \\
&+ \left( 1 - 5e^2 + \frac{63}{8} e^4 - \frac{155}{36} e^6 \right) k_2 \sin |\epsilon_{2200}| \operatorname{sgn} \left( n - \dot{\theta} \right) \\
&+ \left( \frac{49}{4} e^2 - \frac{861}{16} e^4 + \frac{21975}{256} e^6 \right) k_2 \sin |\epsilon_{2201}| \operatorname{sgn} \left( 3n - 2\dot{\theta} \right) \\
&+ \left( \frac{289}{4} e^4 - \frac{1955}{6} e^6 \right) k_2 \sin |\epsilon_{2202}| \operatorname{sgn} \left( 2n - \dot{\theta} \right) \\
&\left. + \frac{714025}{2304} e^6 k_2 \sin |\epsilon_{2203}| \operatorname{sgn} \left( 5n - 3\dot{\theta} \right) \right] + O(e^8 \epsilon) + O(i^2 \epsilon) \quad . \quad (120)
\end{aligned}$$

Here each term changes its sign on crossing the appropriate resonance. The change of the sign takes place smoothly, as the value of the term goes through zero – this can be seen from formula (95) and from the fact that the tidal mode  $\omega_{lmpq}$  vanishes in the  $lmpq$  resonance.

Expressions for  $\tilde{\mathcal{T}}_{(lmp)=(220)}$ ,  $\tilde{\mathcal{T}}_{(lmp)=(311)}$ , and  $\tilde{\mathcal{T}}_{(lmp)=(330)}$  are given by formulae (289 - 291) in Appendix I. Although the average of the short-period part of the torque vanishes, this part does contribute to dissipation. Oscillating torques contribute also to variations of the surface of the tidally-distorted primary, the latter fact being of importance in laser-ranging experiments.

Whether the short-period torque may or may not influence also the process of entrapment is worth exploring numerically. The reason why this issue is raised is that the frequencies  $n(q-j)$  of the components of the short-period torque are integers of  $n$  and thus are commensurate with the spin rate  $\dot{\theta}$  near an  $A/B$  resonance,  $A$  and  $B$  being integer. It may be especially interesting to check the role of this torque when  $q-j=1$  and  $A/B=N$  is integer.

The hypothetical role of the short-period torque in the entrapment and libration dynamics has never been discussed so far, as the previous studies employed expressions for the tidal torque, which were obtained through averaging over the period of the secondary's orbital motion.

### 8.3 Librations

Consider a tidally-perturbed body caught in a  $A:B$  resonance with its tide-raising companion,  $A$  and  $B$  being integer. Then the spin rate of the body is

$$\dot{\theta} = \frac{A}{B} n + \dot{\psi} \quad , \quad (121)$$

where the physical-libration angle is

$$\psi = -\psi_0 \sin(\omega_{PL} t) \quad , \quad (122)$$



$\omega_{PL}$  being the physical-libration frequency. The oscillating tidal torque exerted on the body is comprised of the modes

$$\omega_{lmpq} = (l - 2p + q) n - m \dot{\theta} = \left( l - 2p + q - \frac{A}{B} m \right) n - m \dot{\psi} . \quad (123)$$

In those  $lmpq$  terms for which the combination  $l - 2p + q - \frac{A}{B} m$  is not zero, the small quantity  $-m\dot{\psi}$  may be neglected.<sup>17</sup> The remaining terms will pertain to the geometric libration. The phase lags will be given by the standard formula  $\epsilon_{lmpq} = \omega_{lmpq} \Delta t_{lmpq}$ .

In those  $lmpq$  terms for which the combination  $l - 2p + q - \frac{A}{B} m$  vanishes, the physical-libration input  $-m\dot{\psi}$  is the only one left. Accordingly, the multiplier  $\sin \left[ \left( v_{lmpq}^* - m\theta^* \right) - \left( v_{lmpq} - m\theta \right) \right]$  in the  $lmpq$  term of the tidal torque will reduce to  $\sin [-m(\psi^* - \psi)] \approx -m\dot{\psi} \Delta t = m\psi_0 \omega_{PL} \Delta t \cos \omega_{PL} t$ . Here the time lag  $\Delta t$  is the one corresponding to the physical-libration frequency  $\omega_{PL}$  which may be *very different* from the usual tidal frequencies for nonsynchronous rotation – see Williams & Efroimsky (2012) for a comprehensive discussion.

## 9 Marking the minefield

The afore-presented expressions for the secular and purely short-period parts of the tidal torque look cumbersome when compared to the compact and elegant formulae employed in the literature hitherto. It will therefore be important to explain why those simplifications are impractical.

### 9.1 Perils of the conventional simplification

Insofar as the quality factor is large and the lag is small (i.e., insofar as  $\sin \epsilon$  can be approximated with  $\epsilon$ ), our expression (282a) assumes a simpler form:

$$^{(Q > 10)} \overline{\mathcal{T}}_{l=2} = \frac{3}{2} G M_{sec}^2 R^5 a^{-6} k_2 \sum_{q=-3}^3 G_{20q}^2(e) \epsilon_{220q} + O(e^6 \epsilon) + O(i^2 \epsilon) + O(\epsilon^3) , \quad (124)$$

where the error  $O(\epsilon^3)$  originates from  $\sin \epsilon = \epsilon + O(\epsilon^3)$ .

The simplification conventionally used in the literature ignores the frequency-dependence of the Love number and attributes the overall frequency-dependence to the lag. It also ignores the difference between the tidal lag  $\epsilon$  and the lag in the material,  $\delta$ . This way, the conventional simplification makes  $\epsilon$  obey the scaling law (79b). At this point, most authors also set  $\alpha = -1$ . Here we shall explore this approach, though shall keep  $\alpha$  arbitrary. From the formula<sup>18</sup>

$$\Delta t_{lmpq} = \mathcal{E}^{-\alpha} \chi_{lmpq}^{-(\alpha+1)} \quad (125)$$

derived by Efroimsky & Lainey (2007) in the said approach, we see that the time lags are related to the principal-frequency lag  $\Delta t_{2200}$  via:

$$\Delta t_{lmpq} = \Delta t_{2200} \left( \frac{\chi_{2200}}{\chi_{lmpq}} \right)^{\alpha+1} . \quad (126)$$

<sup>17</sup> The physical-libration input  $-m\dot{\alpha}$  may be neglected in the expression for  $\omega_{lmpq}$  even when the magnitude of the physical libration is comparable to that of the geometric libration (as in the case of Phobos).

<sup>18</sup> Let  $\frac{1}{\sin \epsilon} = (\mathcal{E} \chi)^\alpha$ , where  $\mathcal{E}$  is an empirical parameter of the dimensions of time, while  $\epsilon$  is small enough, so  $\sin \epsilon \approx \epsilon$ . In combination with  $\epsilon_{lmpq} \equiv \omega_{lmpq} \Delta t_{lmpq}$  and  $\chi_{lmpq} = |\omega_{lmpq}|$ , these formulae entail (125).

When the despinning is still going on and  $\dot{\theta} \gg n$ , the corresponding phase lags are:

$$\epsilon_{lmpq} \equiv \Delta t_{lmpq} \omega_{lmpq} = - \Delta t_{2200} \chi_{lmpq} \left( \frac{\chi_{2200}}{\chi_{lmpq}} \right)^{\alpha+1} = - \epsilon_{2200} \left( \frac{\chi_{2200}}{\chi_{lmpq}} \right)^{\alpha}, \quad (127)$$

which helps us to cast the secular part of the torque into the following convenient form:<sup>19</sup>

$$\begin{aligned} {}^{(Q>10)}\overline{\mathcal{T}}_{l=2} &= \mathcal{Z} \left[ -\dot{\theta} \left( 1 + \frac{15}{2} e^2 + \frac{105}{4} e^4 + O(e^6) \right) \right. \\ &\quad \left. + n \left( 1 + \left( \frac{15}{2} - 6\alpha \right) e^2 + \left( \frac{105}{4} - \frac{363}{8} \alpha \right) e^4 + O(e^6) \right) \right] + O(i^2/Q) + O(Q^{-3}) + O(\alpha e^2 Q^{-1} n/\dot{\theta}) \end{aligned} \quad (128a)$$

$$\approx \mathcal{Z} \left[ -\dot{\theta} \left( 1 + \frac{15}{2} e^2 \right) + n \left( 1 + \left( \frac{15}{2} - 6\alpha \right) e^2 \right) \right], \quad (128b)$$

where the overall factor reads as:

$$\mathcal{Z} = \frac{3 G M_{sec}^2 k_2 \Delta t_{2200}}{R} \frac{R^6}{a^6} = \frac{3 n^2 M_{sec}^2 k_2 \Delta t_{2200}}{(M_{prim} + M_{sec})} \frac{R^5}{a^3} = \frac{3 n M_{sec}^2 k_2}{Q_{2200} (M_{prim} + M_{sec})} \frac{R^5}{a^3} \frac{n}{\chi_{2200}}, \quad (129)$$

$M_{prim}$  and  $M_{sec}$  being the primary's and secondary's masses.<sup>20</sup> Dividing (129) by the primary's principal moment of inertia  $\xi M_{primary} R^2$ , we obtain the contribution that this component of the torque brings into the angular deceleration rate  $\ddot{\theta}$ :

$$\begin{aligned} \ddot{\theta} &= \mathcal{K} \left\{ -\dot{\theta} \left[ 1 + \frac{15}{2} e^2 + \frac{105}{4} e^4 + O(e^6) \right] + \right. \\ &\quad \left. n \left[ 1 + \left( \frac{15}{2} - 6\alpha \right) e^2 + \left( \frac{105}{4} - \frac{363}{8} \alpha \right) e^4 + O(e^6) \right] \right\} + O(i^2/Q) + O(Q^{-3}) + O(\alpha e^2 Q^{-1} n/\dot{\theta}) \end{aligned} \quad (130a)$$

$$\approx \mathcal{K} \left[ -\dot{\theta} \left( 1 + \frac{15}{2} e^2 \right) + n \left( 1 + \left( \frac{15}{2} - 6\alpha \right) e^2 \right) \right], \quad (130b)$$

the factor  $\mathcal{K}$  being given by

$$\mathcal{K} \equiv \frac{\mathcal{Z}}{\xi M_{prim} R^2} = \frac{3 n^2 M_{sec}^2 k_2 \Delta t_{2200}}{\xi M_{prim} (M_{prim} + M_{sec})} \frac{R^3}{a^3} = \frac{3 n M_{sec}^2 k_2}{\xi Q_{2200} M_{prim} (M_{prim} + M_{sec})} \frac{R^3}{a^3} \frac{n}{\chi_{2200}}, \quad (131)$$

where  $\xi$  is a multiplier emerging in the expression  $\xi M_{primary} R^2$  for the primary's principal moment of inertia ( $\xi = 2/5$  for a homogeneous sphere).

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<sup>19</sup> For  $\dot{\theta} \gg 2n$ , all the modes  $\omega_{220q}$  are negative, so  $\omega_{220q} = -\chi_{220q}$ . Then, keeping in mind that  $n/\dot{\theta} \ll 1$ , we process (127), for  $q = 1$ , like

$$- \Delta t_{2200} \chi_{2200} \left( \frac{\chi_{2200}}{\chi_{2201}} \right)^{\alpha} = - \Delta t_{2200} 2|n - \dot{\theta}| \left( \frac{2|n - \dot{\theta}|}{|-2\dot{\theta} + 3n|} \right)^{\alpha} = - \Delta t_{2200} 2(\dot{\theta} - n) \left[ 1 + \frac{\alpha}{2} \frac{n}{\dot{\theta}} + O((n/\dot{\theta})^2) \right],$$

and similarly for other  $q \neq 0$ , and then plug the results into (124). This renders us (128).

<sup>20</sup> To arrive at the right-hand side of (129), we recalled that  $\chi_{lmpq} \Delta t_{lmpq} = |\epsilon_{lmpq}|$  and that  $Q_{lmpq}^{-1} = |\epsilon_{lmpq}| + O(\epsilon^3) = |\epsilon_{lmpq}| + O(Q^{-3})$ , according to formula (12).

In the special case of  $\alpha = -1$ , the above expressions enjoy agreement with the appropriate result stemming from the corrected MacDonald model – except that our (128) and (130) contain  $\Delta t_{2200}$ ,  $Q_{2200}$ , and  $\chi_{2200}$  instead of  $\Delta t$ ,  $Q$ , and  $\chi$  standing in formulae (44 - 47) from Williams & Efroimsky (2012). Formula (130b) tells us that the secular part of the tidal torque vanishes for

$$\dot{\theta} - n = -6 n e^2 \alpha, \quad (132)$$

which coincides, for  $\alpha = -1$ , with the result obtained in Rodríguez et al. (2008, eqn. 2.4) and Williams & Efroimsky (2012, eqn. 49). This coincidence however should not be taken at its face value, because it is *occasional* or, possibly better to say, *exceptional*.

Formulae (128 - 129) were obtained by insertion of the expressions for the eccentricity functions and the phase lags into (124), and by assuming that  $n \ll |\dot{\theta}|$ . The latter caveat is a crucial element, not to be overlooked by the users of formulae (128 - 129) and of their corollary (130 - 131) for the tidal deceleration rate.

The case of  $\alpha = -1$  is special, in that it permits derivation of (128 - 132) without assuming that  $n \ll |\dot{\theta}|$ . However for  $\alpha > -1$  the condition  $n \ll |\dot{\theta}|$  remains mandatory, so formulae (128 - 131) become *inapplicable* when  $\dot{\theta}$  reduces to values of about several  $n$ .

Although formulae (128a) and (130a) contain an absolute error  $O(\alpha e^2 Q^{-1} n / \dot{\theta})$ , this does not mean that for  $\dot{\theta}$  comparable to  $n$  the absolute error becomes  $O(\alpha e^2 Q^{-1})$  and the relative one becomes  $O(\alpha e^2)$ . In reality, for  $\dot{\theta}$  comparable to  $n$ , *the entire approximation falls apart*, because formulae (126 - 127) were derived from expression (125), which is valid for  $Q \gg 1$  only (unless  $\alpha = -1$ ). So these formulae become inapplicable in the vicinity of a commensurability. By ignoring this limitation, one can easily encounter unphysical paradoxes.<sup>21</sup>

Thence, in all situations, except for the unrealistic rheology  $\alpha = -1$ , limitations of the approximation (128 - 131) should be kept in mind. This approximation remains acceptable for  $n \ll |\dot{\theta}|$ , but becomes misleading on approach to the physically-interesting resonances.

## 9.2 An oversight in a popular formula

The form in which our approximation (128 - 131) is cast may appear awkward. The formula for the despinning rate  $\ddot{\theta}$  is written as a function of  $\dot{\theta}$  and  $n$ , multiplied by the overall factor  $\mathcal{K}$ . This form would be reasonable, were  $\mathcal{K}$  a constant. That this is not the case can be seen from the presence of the multiplier  $\frac{n}{\chi_{2200} 2|\dot{\theta} - n|}$  on the right-hand side of (131).

Still, when written in this form, our result is easy to juxtapose with an appropriate formula from Correia & Laskar (2004, 2009). There, the expression for the despinning rate looks similar to ours, up to an important detail: the overall factor is a constant, because it lacks the said multiplier  $\frac{n}{\chi_{2200}}$ . The multiplier was lost in those two papers, because the quality factor was introduced there as  $1/(n \Delta t)$ , see the line after formula (9) in Correia & Laskar (2009). In reality, the quality factor  $Q$  should, of course, be a function of the tidal frequency  $\chi$ , because  $Q$  serves the purpose of describing the tidal damping at this frequency. Had the quality factor been taken as  $1/(\chi \Delta t)$ , it would render the corrected MacDonald model ( $\alpha = -1$ ), and the missing multiplier would be there. Being unphysical,<sup>22</sup> the model is mathematically convenient, because it enables one to write down the secular part of the torque as one expression, avoiding

<sup>21</sup> For example, in the case of  $\alpha > -1$ , formulae (125 - 127) render infinite values for  $\Delta t_{lmpq}$  and  $\epsilon_{lmpq}$  on crossing the commensurability, i.e., when  $\omega_{lmpq}$  goes through zero.

<sup>22</sup> To be exact, the model is unphysical everywhere except in the closest vicinity of the resonance – see formulae (92 - 94).

the expansion into a series (Williams & Efroimsky 2012). The model was pioneered by Singer (1968) and employed by Mignard (1979, 1980), Hut (1981) and other authors.

Interestingly, in the special case of the 3:2 spin-orbit resonance, we have  $\chi = n$ . Still, the difference between  $\chi$  and  $n$  in the vicinity of the resonance may alter the probability of entrapment of Mercury into this rotation mode. The difference between  $\chi$  and  $n$  becomes even more considerable near the other resonances of interest. So the probabilities of entrapment into those resonances must be recalculated.

## 10 Conclusions

The goal of this paper was to lay the ground for a reliable model of tidal entrapment into spin-orbital resonances. To this end, we approached the tidal theory from the first principles of solid-state mechanics. Starting from the expression for the material's compliance in the time domain, we derived the frequency-dependence of the Fourier components of the tidal torque. The other torque, one caused by the triaxiality of the rotator, is not a part of this study and will be addressed elsewhere.

- We base our work on the Andrade rheological model, providing arguments in favour of its applicability to the Earth's mantle, and therefore, very likely, to other terrestrial planets and moons. The model is also likely to apply to the icy moons (Castillo-Rogez et al. 2011). We have reformulated the model in terms of a characteristic anelastic timescale  $\tau_A$  (the Andrade time). The ratio of the Andrade time to the viscoelastic Maxwell time,  $\zeta = \tau_A/\tau_M$ , serves as a dimensionless free parameter of the rheological model.

The parameters  $\tau_A$ ,  $\tau_M$ ,  $\zeta$  cannot be regarded constant, though their values may be changing very slowly over vast bands of frequency. The shapes of these frequency-dependencies may depend on the dominating dissipation mechanisms and, thereby, on the magnitude of the load, as different damping mechanisms get activated under different loads.

The main question here is whether, in the low-frequency limit, anelasticity becomes much weaker than viscosity. (That would imply an increase of  $\tau_A$  and  $\zeta$  as the tidal frequency  $\chi$  goes down.) The study of ices under weak loads, with friction caused mainly by lattice diffusion (Castillo-Rogez et al. 2011, Castillo-Rogez & Choukroun 2010) has not shown such a decline of anelasticity. However, Karato & Spetzler (1990) point out that it should be happening in the Earth's mantle, where the loads are much higher and damping is caused mainly by unpinning of dislocations. According to *Ibid.*, in the Earth, the decrease of the role of anelasticity happens abruptly as the frequency falls below the threshold  $\chi_0 \sim 1 \text{ yr}^{-1}$ . We then may expect a similar switch in the other terrestrial planets and the Moon, though there the threshold may be different as it is sensitive to the temperature of the mantle. The question, though, remains if this statement is valid also for the small bodies, in which the tidal stresses are weaker and dissipation is dominated by lattice diffusion.

- By direct calculation, we have derived the frequency dependencies of the factors  $k_l \sin \epsilon_l$  emerging in the tidal theory. Naturally, the obtained dependencies of these factors upon the tidal frequency  $\chi_{lmpq}$  (or, to put it more exactly, upon the tidal mode  $\omega_{lmpq}$ ) mimic the frequency-dependence of the imaginary part of the complex compliance. They scale as  $\sim \chi^{-\alpha}$  with  $0 < \alpha < 1$ , at higher frequencies; and look as  $\sim \chi^{-1}$  at lower frequencies. However in the zero-frequency limit the factors  $k_l \sin \epsilon_l$  demonstrate a behavior inherent in the tidal lagging and absent in the case of lagging in a sample: in a close vicinity of the zero frequency, these factors (and the appropriate components of the tidal torque) become linear in the frequency. This way,  $k_l \sin \epsilon_l$  first reaches a finite maximum, then decreases continuously to nil as the frequency approaches to zero, and then changes its sign. So

the resonances are crossed continuously, with neither the tidal torque nor the tidal force diverging there. For example, the leading term of the torque vanishes at the synchronous orbit.

This continuous traversing of resonances was pointed out in a heuristic manner by Efroimsky & Williams (2009). Now we have derived this result directly from the expression for the compliance of the material of the rotating body. Our derivation, however, has a problem in it: the frequency, below which the factors  $k_l \sin \epsilon_l$  and the appropriate components of the torque change their frequency-dependence to linear, is implausibly low (lower than  $10^{-10}$  Hz, if we take our formulae literally). The reason for this mishap is that in our formulae we kept using the known value of the Maxwell time  $\tau_M$  all the way down to the zero frequency. Possible changes of the viscosity and, accordingly, of the Maxwell time in the zero-frequency limit may broaden this region of linear dependence.

- We have offered an explanation of the unexpected frequency-dependence of dissipation in the Moon, discovered by LLR. The main point of our explanation is that the LLR measures the *tidal* dissipation whose frequency-dependence is different from that of the *seismic* dissipation. Specifically, the “wrong” sign of the exponent in the power dissipation law may indicate that the frequencies at which tidal friction was observed were below the frequency at which the lunar  $k_2 \sin \epsilon_2$  has its peak. Taken the relatively high frequencies of observation (corresponding to periods of order month to year), this explanation can be accepted only if the lunar mantle has a low mean viscosity. This may be the case, taken the presumably high concentration of the partial melt in the low mantle.
- We have developed a detailed formalism for the tidal torque, and have singled out its oscillating component.

The studies of entrapment into spin-orbit resonances, performed in the past, took into account neither the afore-mentioned complicated frequency-dependence of the torque in the vicinity of a resonance, nor the oscillating part of the torque. We have written down a concise and practical expression for the oscillating part, and have raised the question whether it may play a role in the entrapment and libration dynamics.

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I gladly acknowledge the help and inspiration which I obtained from reading the unpublished preprint by the late Vladimir Churkin (1998). In Appendix C, I present several key results from his preprint.

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# Appendix.

## A Continuum mechanics.

### A celestial-mechanician's survival kit

Appendix A offers an extremely short introduction into continuum mechanics. The standard material, which normally occupies large chapters in books, is compressed into several pages.

Subsection A.1 presents the necessary terms and definitions. Subsections A.2 explains the basic concepts employed in the theory of stationary deformation, while subsection A.3 explains extension of these methods to creep. These subsections also demonstrate the great benefits stemming from the isotropy and incompressibility assumptions. Subsection A.4 introduces viscosity, while subsection A.5 offers an example of how elasticity, viscosity get combined with hereditary reaction, into one expression. Subsection A.6 renders several simple examples.

#### A.1 Glossary

We start out with a brief guide elucidating the main terms employed in continuum mechanics.

- *Rheology* is the science of deformation and flow.
- *Elasticity*: This is the most trivial reaction – instantaneous, linear, and fully reversible after stressing is turned off.
- *Anelasticity*: While still linear, this kind of deformation is not necessarily instantaneous, and can demonstrate “memory”, both under loading and when the load is removed. Importantly, the term *anelasticity* always implies reversibility: though with delay, the original shape is restored. Thus an anelastic material can assume two equilibrium states: one is the unstressed state, the other being the long-term relaxed state. Anelasticity is characterised by the difference in strain between the two states. It is also characterised by a relaxation time between these states, and by its inverse – the frequency at which relaxation is most effective. The Hohenemser-Prager model, also called SAS (Standard Anelastic Solid), renders an example of anelastic behaviour.

Anelasticity is an example of but not synonym to *hereditary reaction*. The latter includes also those kinds of delayed deformation, which are irreversible.

- *Inelasticity*: This term embraces *all* kinds of irreversible deformation, i.e., deformation which stays, fully or partially, after the load is removed.
- *Unelasticity* (= *Nonelasticity*): These terms are very broad, in that they embrace any behaviour which is not elastic. In the literature, these terms are employed both for recoverable (anelastic) and unrecoverable (inelastic) deformations.
- *Plasticity*: Some authors simply state that plastic deformation is deformation which is irreversible – a very broad and therefore useless definition which makes plasticity sound like a synonym to inelasticity.

More precisely, plastic is a stepwise behaviour: no strain is observed until the stress  $\sigma$  reaches a threshold value  $\sigma_Y$  (called yield strength), while a steady flow begins as the stress reaches the said threshold. Plasticity can be either perfect (when deformation is going on without any increase in load) or with hardening (when increasingly higher stresses are needed to sustain the flow). It is always irreversible.

In real life, plasticity shows itself in combination with elasticity or/and viscosity. Models describing these types of behaviour are called elastoplastic, viscoplastic, and viscoelastoplastic. They are all inelastic, in that they describe unrecoverable changes of shape.

- *Viscosity*: Another example of inelastic, i.e., irreversible behaviour. A viscous stress is proportional to the time derivative of the viscous strain.
- *Viscoelasticity*: The term is customarily applied to all situations where both viscous and elastic (but not plastic) reactions are observed. One may then expect that the equations interrelating the viscoelastic stress to the strain would contain only the viscosity coefficients and the elastic moduli. However this is not necessarily true, as some other empirical constants may show up. For example, the Andrade model (81) contains an elastic term, a viscous term, and an extra term responsible for a hereditary reaction (the “memory”). Despite the presence of that extra term, the Andrade creep is still regarded viscoelastic. So it should be understood that viscoelasticity is, generally, more than just viscosity combined with elasticity. One might christen such deformations “viscoelastohereditary”, but such a term would sound awkward.<sup>23</sup>

On many occasions, complex viscoelastic models can be illustrated with an infinite set of viscous and elastic elements. These serve to interpret the hereditary terms as manifestations of viscosity and elasticity only. While illustrative, these schemes with dashpots and springs have their limitations and should not be taken too literally. In some (not all) situations, the hereditary terms may be interpreted as time-dependent additions to the steady-state viscosity coefficient, the Andrade model being an example of such situation.

- *Viscoplasticity*: These are all models wherein both viscosity and plasticity are present in some combination. In these situations, higher stresses have to be applied to increase the deformation rate. Just as in the case of viscoelasticity, viscoplastic models may, in principle, incorporate extra terms standing for hereditary reaction.
- *Elastoviscoplasticity* (= *Viscoelastoplasticity*): The same as above, though with elasticity present.
- *Hereditary reaction*: While the term is self-explanatory, it would be good to limit its use to effects other than viscosity. In expression (46) for the stress through strain, the distinction between the viscous and hereditary reactions is clear: while the viscous part of the stress is rendered instantaneously by the delta-function term of the kernel, the hereditary reaction is obtained through integration. In expression (40) for the strain through stress, though, the viscous part shows up, under the integral, in sum with the other delayed terms – see, for example, the Andrade model (81). This is one reason for which we shall use the term *hereditary reaction* in application to delayed behaviour different from the pure viscosity. Another reason is that viscous flow is always irreversible, while a hereditary reaction may be either irreversible (inelastic) or reversible (anelastic).
- *Creep*: Widely accepted is the convention that this term signifies a slow-rate deformation under loads below the yield strength  $\sigma_Y$ .

Numerous authors, though, use the oxymoron *plastic creep*, thereby extending the applicability realm of the word *creep* to *any* slow deformation.

Here we shall understand creep in the former sense, i.e., with no plasticity involved.

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<sup>23</sup> Sometimes the term *elastohereditary* is used, but not *viscoelastohereditary* or *elastoviscohereditary*.

It would be important to distinguish between viscoelastic deformations, on the one hand, and viscoplastic (or, properly speaking, viscoelastoplastic) deformations on the other hand. Plasticity shows itself at larger stresses and is, typically, nonlinear. It comes into play when the linearity assertion fails. For most minerals, this happens when the strain approaches the threshold of  $10^{-6}$ . Although it is possible that this threshold is transcended in some satellites (for example, in the deeper layers of the Moon), we do not address plasticity in this paper.

## A.2 Stationary linear deformation of isotropic incompressible media

In the linear approximation, the tensor of elastic stress,  $\overset{(e)}{\mathbb{S}}$ , is proportional to the differences in displacement of the neighbouring elements of the medium. These differences are components of the *tensor gradient*  $\nabla \otimes \mathbf{u}$ , where  $\mathbf{u}$  is the displacement vector.

The tensor gradient can be decomposed, in an invariant way, into its symmetric and antisymmetric parts:

$$\nabla \otimes \mathbf{u} = \frac{1}{2} \left[ (\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^T \right] + \frac{1}{2} \left[ (\nabla \otimes \mathbf{u}) - (\nabla \otimes \mathbf{u})^T \right] . \quad (133)$$

The decomposition being invariant, the two parts should contribute into the stress independently, at least in the linear approximation. However, as well known (Landau & Lifshitz 1986), the antisymmetric part of (133) describes the displacement of the medium as a whole and thus brings nothing into the stress. This is why the linear dependence is normally written as

$$\overset{(e)}{\mathbb{S}} = \mathbb{B} \mathbb{U} , \quad (134)$$

where  $\mathbb{B}$  is a four-dimensional tensor having  $3^4 = 81$  components, while the strain tensor

$$\mathbb{U} \equiv \frac{1}{2} \left[ (\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^T \right] \quad (135)$$

is the symmetric part of the tensor gradient. Its components are related to the displacement vector  $\mathbf{u}$  through  $u_{\alpha\beta} \equiv \frac{1}{2} \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)$ .

Although the matrix  $\mathbb{B}$  is comprised of 81 empirical constants, in isotropic materials the description can be reduced to two constants only. To see this, recall that the expansion of the strain into a part with trace and a traceless part,  $\mathbb{U} = \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} + \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right)$ , is invariant. Here the trace of  $\mathbb{U}$  is denoted with  $\text{Sp} \mathbb{U} \equiv u_{\alpha\alpha}$ , summation over repeated indices being implied. The notation  $\mathbb{I}$  stands for the unity matrix consisting of elements  $\delta_{\gamma\nu}$ .

In an isotropic medium, the elastic stress must be invariantly expandable into parts proportional to the afore-mentioned parts of the strain. The first part of the stress is proportional, with an empirical coefficient  $3K$ , to the trace part  $\frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U}$  of the strain. The second part of the stress will be proportional, with an empirical coefficient  $2\mu$ , to the traceless part  $\left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right)$  of the strain:

$$\overset{(e)}{\mathbb{S}} = K \mathbb{I} \text{Sp} \mathbb{U} + 2\mu \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) = -p \mathbb{I} + 2\mu \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) \quad (136a)$$

or, in Cartesian coordinates:

$$\sigma_{\gamma\nu} = K \delta_{\gamma\nu} u_{\alpha\alpha} + 2\mu \left( u_{\gamma\nu} - \frac{1}{3} \delta_{\gamma\nu} u_{\alpha\alpha} \right) = -p \delta_{\gamma\nu} + 2\mu \left( u_{\gamma\nu} - \frac{1}{3} \delta_{\gamma\nu} u_{\alpha\alpha} \right) , \quad (136b)$$



where

$$p \equiv -\frac{1}{3} \text{Sp} \mathbb{S} = -K \text{Sp} \mathbb{U} \quad (137)$$

is the hydrostatic pressure. Thus the elastic stress gets decomposed, in an invariant way, as:

$$\mathbb{S}^{(e)} = \mathbb{S}_{volumetric}^{(e)} + \mathbb{S}_{deviatoric}^{(e)} \quad , \quad (138)$$

where the *volumetric elastic stress* is given by

$$\mathbb{S}_{volumetric}^{(e)} \equiv K \mathbb{I} \text{Sp} \mathbb{U} = \mathbb{I} \frac{1}{3} \text{Sp} \mathbb{S} = -p \mathbb{I} \quad , \quad (139)$$

while the *deviatoric elastic stress* is:

$$\mathbb{S}_{deviatoric}^{(e)} \equiv 2\mu \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{U} \right) \quad . \quad (140)$$

Inverse to (136a - 136b) are the following expressions for the strain tensor:

$$\mathbb{U} = \frac{1}{9K} \mathbb{I} \text{Sp} \mathbb{S}^{(e)} + \frac{1}{2\mu} \left( \mathbb{S}^{(e)} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{S}^{(e)} \right) \quad (141a)$$

and

$$u_{\gamma\nu} = \frac{1}{9K} \delta_{\gamma\nu} \sigma_{\alpha\alpha}^{(e)} + \frac{1}{2\mu} \left( \sigma_{\gamma\nu}^{(e)} - \frac{1}{3} \delta_{\gamma\nu} \sigma_{\alpha\alpha}^{(e)} \right) \quad , \quad (141b)$$

where the term with trace,  $\frac{1}{9K} \mathbb{I} \text{Sp} \mathbb{S}^{(e)}$ , is called the *volumetric strain*, while the traceless term,  $\frac{1}{2\mu} \left( \mathbb{S}^{(e)} - \frac{1}{3} \mathbb{I} \text{Sp} \mathbb{S}^{(e)} \right)$ , is named the *deviatoric strain*. The quantity  $K$  is called the *bulk modulus*, while  $\mu$  is called the *shear modulus*.

Expressions (136) trivially entail the following interrelation between traces:

$$\text{Sp} \mathbb{S}^{(e)} = 3K \text{Sp} \mathbb{U} \quad \text{or, in terms of components:} \quad \sigma_{\alpha\alpha}^{(e)} = 3K u_{\alpha\alpha} \quad . \quad (142)$$

As demonstrated in many books (e.g., in Landau & Lifshitz 1986), the trace of the strain is equal to the relative variation of the volume, experienced by the material subject to deformation:  $u_{\alpha\alpha} = \nabla \cdot \mathbf{u} = \frac{dV' - dV}{dV}$ , where  $\mathbf{u}$  is the displacement vector. In the no-compressibility approximation, the trace of the strain and, according to (142), that of the stress become zero. Then, in the said approximation, the hydrostatic pressure (137) and the volumetric elastic stress (139) become nil, and all we are left with is the deviatoric elastic stress (and, accordingly, the deviatoric part of the strain). Formulae (136) and (141) get simplified to

$$\mathbb{S}^{(e)} = 2\mu \mathbb{U} \quad , \quad \text{which is the same as} \quad \sigma_{\gamma\nu}^{(e)} = 2\mu u_{\gamma\nu} \quad , \quad (143)$$

and to

$$2\mathbb{U} = J \mathbb{S}^{(e)} \quad , \quad \text{which is the same as} \quad 2u_{\gamma\nu} = J \sigma_{\gamma\nu}^{(e)} \quad , \quad (144)$$

the quantity  $J \equiv 1/\mu$  being called the *compliance* of the material.

### A.3 Evolving linear deformation of isotropic incompressible isotropic media. Hereditary reaction

Equations (134 - 144) were written for static deformation, so each of these equations can be assumed to connect the strain and the elastic stress taken at the same instant of time (for a static deformation their values stay constant anyway).

Extension of this machinery is needed when one wants to describe evolving deformation of materials with “memory”. Thence the four-dimensional tensor  $\mathbb{B}$  becomes a linear operator  $\tilde{\mathbb{B}}$  acting on the strain tensor function as a whole. To render the value of the stress at time  $t$ , the operator will “consume” as arguments all the values of strain over the interval  $t' \in (-\infty, t]$ :

$$\overset{(h)}{\mathbb{S}}(t) = \tilde{\mathbb{B}}(t) \mathbb{U} \quad . \quad (145)$$

Thus  $\tilde{\mathbb{B}}$  will be an integral operator, with integration going from  $t' = -\infty$  through  $t' = t$ .

In the static case, the linearity guaranteed elasticity, i.e., the ability of the body to regain its shape after the loading is turned off: no stress yields no strain. In a more general situation of materials with “memory”, this ability is no longer retained, as the material may demonstrate *creep*. This is why, in this section, the stress is called *hereditary* and is denoted with  $\overset{(h)}{\mathbb{S}}$ .

Just as in the stationary case, we wish the properties of the medium to remain isotropic. As the decomposition of the strain into the trace and traceless parts remains invariant at each moment of time, these two parts will, separately, generate the trace and traceless parts of the hereditary stress in an isotropic medium. This means that, in such media, the four-dimensional tensor operator  $\tilde{\mathbb{B}}$  gets reduced to two scalar linear operators  $\tilde{K}$  and  $\tilde{\mu}$ :

$$\overset{(h)}{\mathbb{S}} = \overset{(h)}{\mathbb{S}}_{\text{volumetric}} + \overset{(h)}{\mathbb{S}}_{\text{deviatoric}} = \tilde{K} \mathbb{I} \text{Sp } \mathbb{U} + 2\tilde{\mu} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) \quad , \quad (146)$$

where both  $\tilde{K}$  and  $\tilde{\mu}$  are integral operators acting on the tensor function  $u_{\gamma\nu}(t')$  as a whole, i.e., with integration going from  $t' = -\infty$  through  $t' = t$ .

If we also assume that, under evolving load, the medium stays incompressible, the trace of the strain,  $u_{\alpha\alpha}$ , will stay zero. An operator generalisation of (142) now reads:  $\sigma_{\alpha\alpha}(t) = 3\tilde{K}(t)u_{\alpha\alpha}$ . Under a reasonable assumption of  $\sigma_{\alpha\alpha}$  being nil in the distant past, this integral operator renders  $\sigma_{\alpha\alpha} = 0$  at all times. This way, in a medium that is both isotropic and incompressible, we have:

$$\mathbb{U} = \mathbb{U}_{\text{deviatoric}} \quad \text{and, accordingly:} \quad \overset{(h)}{\mathbb{S}} = \overset{(h)}{\mathbb{S}}_{\text{deviatoric}} \quad . \quad (147)$$

Then the time-dependent analogues to formulae (143) and (144) will be:

$$\overset{(h)}{\mathbb{S}}(t) = 2\tilde{\mu}(t)\mathbb{U} \quad (148)$$

and

$$2\mathbb{U}(t) = \hat{J}(t) \overset{(h)}{\mathbb{S}} \quad , \quad (149)$$

where the compliance  $\hat{J}$ , too, has been promoted to operatorship and crowned with a caret.

Formula (148) tells us that in a medium, which is both isotropic and incompressible, relation between the stress and strain tensors can be described with one scalar integral operator  $\tilde{\mu}$  only, the complementary operator  $\hat{J}$  being its inverse. (Here the adjective “scalar” does not imply multiplication with a scalar. It means that the operator preserves its functional form under a change of coordinates.)

Below we shall bring into the picture also the viscous component of the stress, a component related to the strain through a four-dimensional tensor whose  $3^4 = 81$  components are

differential operators. In that case too, the isotropy of the medium will enable us to reduce the 81-component tensor operator to two differential operators transforming as scalars. Besides, the incompressibility of the medium makes the viscous stress traceless. Thus it will turn out that, in an isotropic and incompressible medium, the viscous component of the stress can be described by only one scalar differential operator – much like the elastic and hereditary parts of the stress. (Once again, “scalar” means: indifferent to coordinate transformations.)

Eventually, the elastic, hereditary, and viscous deformations will be united under the auspices of a general viscoelastic formalism. In an isotropic medium, this combined formalism will be reduced to two integro-differential operators only. In a medium which is both isotropic and incompressible, the formalism will be reduced to only one scalar integro-differential operator.

## A.4 The viscous stress

While the elastic stress  $\overset{(e)}{\mathbb{S}}$  is linear in the strain, the viscous stress  $\overset{(v)}{\mathbb{S}}$  is linear in the first derivatives of the components of the velocity with respect to the coordinates:

$$\overset{(v)}{\mathbb{S}} = \mathbb{A} (\nabla \otimes \mathbf{v}) \quad (150)$$

where  $\mathbb{A}$  is the so-called viscosity tensor,  $\nabla \otimes \mathbf{v}$  is the tensor gradient of the velocity. The velocity of a fluid parcel relative to its average position is connected to the displacement vector  $\mathbf{u}$  through  $\mathbf{v} = d\mathbf{u}/dt$ .

The tensor gradient of the velocity can be expanded, in an invariant way, into its antisymmetric and symmetric parts:

$$\nabla \otimes \mathbf{v} = \Omega + \mathbb{E} \quad , \quad (151)$$

where the antisymmetric part is furnished by the *vorticity tensor*

$$\Omega \equiv \frac{1}{2} \left[ (\nabla \otimes \mathbf{v}) - (\nabla \otimes \mathbf{v})^T \right] \quad , \quad (152)$$

while the symmetric part is given by the *rate-of-shear tensor*

$$\mathbb{E} \equiv \frac{1}{2} \left[ (\nabla \otimes \mathbf{v}) + (\nabla \otimes \mathbf{v})^T \right] \quad . \quad (153)$$

The latter is obviously related to the strain tensor through

$$\mathbb{E} = \frac{\partial}{\partial t} \mathbb{U} \quad . \quad (154)$$

It can be demonstrated (e.g., Landau & Lifshitz 1987) that the antisymmetric vorticity tensor describes the rotation of the medium as a whole<sup>24</sup> and therefore contributes nothing to the stress.<sup>25</sup> For this reason, the viscous stress can be written as

$$\overset{(v)}{\mathbb{S}} = \mathbb{A} \mathbb{E} = \mathbb{A} \frac{\partial}{\partial t} \mathbb{U} \quad . \quad (155)$$

The matrix  $\mathbb{A}$  is four-dimensional and contains  $3^4 = 81$  components. Just as the matrix  $\mathbb{B}$  emerging in expression (134) for the elastic stress, the matrix  $\mathbb{A}$  can be reduced, in an isotropic

<sup>24</sup> This is why this tensor’s components coincide with those of the angular velocity  $\vec{\omega}$  of the body. For example,  $\Omega_{21} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)$  coincides with  $\omega_3$ .

<sup>25</sup> Since expansion (151) of the tensor gradient into the vorticity and rate-of-shear tensors is invariant, then so is the conclusion about the irrelevance of the vorticity tensor for the stress picture.

medium, to only two empirical constants. To see this, mind that the rate-of-shear tensor can be decomposed, in an invariant manner, into two parts:

$$\mathbb{E} = \frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} + \left( \mathbb{E} - \frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} \right) . \quad (156)$$

where the *rate-of-expansion tensor*

$$\frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} \quad (157)$$

is diagonal and has a trace, while the combination

$$\mathbb{E} - \frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} = \frac{1}{2} \left[ (\nabla \otimes \mathbf{v}) + (\nabla \otimes \mathbf{v})^T \right] - \frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} \quad (158)$$

is symmetric and traceless. These two parts contribute linearly proportional inputs into the stress. The first input is proportional, with an empirical coefficient  $3\zeta$ , to the rate-of-expansion term, while the second input into the stress is proportional, with an empirical coefficient  $2\eta$ , to the symmetric traceless combination:

$$\overset{(v)}{\mathbb{S}} = 3\zeta \frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} + 2\eta \left( \mathbb{E} - \frac{1}{3} \mathbb{I} \nabla \cdot \mathbf{v} \right) = \zeta \frac{\partial}{\partial t} (\mathbb{I} \text{Sp } \mathbb{U}) + 2\eta \frac{\partial}{\partial t} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) . \quad (159)$$

Here we recalled that  $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial t} \nabla \cdot \mathbf{u} = \frac{\partial}{\partial t} u_{\alpha\alpha} = \frac{\partial}{\partial t} \text{Sp } \mathbb{U}$ . Since  $\text{Sp } \mathbb{U}$  is equal to the volume variation  $\frac{dV' - dV}{dV}$  experienced by the material, we see that the first term in (159) is volumetric, the second being deviatoric.

The quantity  $\eta$  is called the *first viscosity* or the *shear viscosity*. The quantity  $\zeta$  is named the *second viscosity* or the *bulk viscosity*.

## A.5 An example of approach to viscoelastic behaviour

In this subsection, we shall consider one possible approach to description of viscoelastic regimes. As we mentioned in subsection A.1, the term *viscoelasticity* covers not only combinations of elasticity and viscosity, but can also include other forms of delayed reaction. So the term *viscoelastic* is customarily used as a substitution for too long a term *viscoelastohereditary*.

One possible approach would be to assume that the elastic, hereditary, and viscous stresses simply sum up, and that each of them is related the the same strain  $\mathbb{U}$ : an

$$\overset{(total)}{\mathbb{S}} = \overset{(e)}{\mathbb{S}} + \overset{(h)}{\mathbb{S}} + \overset{(v)}{\mathbb{S}} = \left( \mathbb{B} + \tilde{\mathbb{B}} + \mathbb{A} \frac{\partial}{\partial t} \right) \mathbb{U} , \quad (160a)$$

or simply

$$\overset{(total)}{\mathbb{S}} = \hat{\mathbb{B}} \mathbb{U} , \quad \text{where } \hat{\mathbb{B}} \equiv \mathbb{B} + \tilde{\mathbb{B}} + \mathbb{A} \frac{\partial}{\partial t} , \quad (160b)$$

where the three operators – the integral operator  $\tilde{\mathbb{B}}$ , the differential operator  $\mathbb{A} \frac{\partial}{\partial t}$ , and the operator of multiplication by matrix  $\mathbb{B}$  – comprise an integro-differential operator  $\hat{\mathbb{B}}$ .

In an isotropic medium, each of the three matrices,  $\tilde{\mathbb{B}}$ ,  $\mathbb{A} \frac{\partial}{\partial t}$ , and  $\mathbb{B}$ , includes two terms only. This happens because in such a medium each of the three parts of the stress gets decomposed invariantly into its deviatoric and volumetric components: The elastic stress becomes:

$$\overset{(e)}{\mathbb{S}} = \overset{(e)}{\mathbb{S}}_{\text{volumetric}} + \overset{(e)}{\mathbb{S}}_{\text{deviatoric}} = 3K \left( \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) + 2\mu \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) , \quad (161)$$

with  $K$  and  $\mu$  being the *bulk elastic modulus* and the *shear elastic modulus*, correspondingly,  $\mathbb{I}$  standing for the unity matrix, and  $\text{Sp}$  denoting the trace of a matrix:  $\text{Sp } \mathbb{U} \equiv \sum_i U_{ii}$ .

The hereditary stress becomes:

$$\overset{(h)}{\mathbb{S}} = \overset{(h)}{\mathbb{S}}_{\text{volumetric}} + \overset{(h)}{\mathbb{S}}_{\text{deviatoric}} = 3\tilde{K} \left( \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) + 2\tilde{\mu} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) , \quad (162)$$

where  $\tilde{K}$  and  $\tilde{\mu}$  are the *bulk-modulus operator* and the *shear-modulus operator*, accordingly.

The viscous stress acquires the form:

$$\overset{(v)}{\mathbb{S}} = \overset{(v)}{\mathbb{S}}_{\text{volumetric}} + \overset{(v)}{\mathbb{S}}_{\text{deviatoric}} = 3\zeta \frac{\partial}{\partial t} \left( \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) + 2\eta \frac{\partial}{\partial t} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) , \quad (163)$$

the quantities  $\zeta$  and  $\eta$  being termed as the *bulk viscosity* and the *shear viscosity*, correspondingly

The term  $\frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U}$  is called the *volumetric* part of the strain, while  $\mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U}$  is called the *deviatoric* part. Accordingly, in expressions (161 - 163) for the stresses, the pure-trace terms are called *volumetric*, the other term being named *deviatoric*.

The total stress, too, can now be split into the total volumetric and the total deviatoric parts:

$$\begin{aligned} \overset{(total)}{\mathbb{S}} &= \overbrace{\left( \overset{(e)}{\mathbb{S}}_{\text{volumetric}} + \overset{(e)}{\mathbb{S}}_{\text{deviatoric}} \right)}^{\overset{(e)}{\mathbb{S}}} + \overbrace{\left( \overset{(v)}{\mathbb{S}}_{\text{volumetric}} + \overset{(v)}{\mathbb{S}}_{\text{deviatoric}} \right)}^{\overset{(v)}{\mathbb{S}}} + \overbrace{\left( \overset{(h)}{\mathbb{S}}_{\text{volumetric}} + \overset{(h)}{\mathbb{S}}_{\text{deviatoric}} \right)}^{\overset{(h)}{\mathbb{S}}} \\ &= \overbrace{\left( \overset{(e)}{\mathbb{S}}_{\text{volumetric}} + \overset{(v)}{\mathbb{S}}_{\text{volumetric}} + \overset{(h)}{\mathbb{S}}_{\text{volumetric}} \right)}^{\mathbb{S}_{\text{volumetric}}} + \overbrace{\left( \overset{(e)}{\mathbb{S}}_{\text{deviatoric}} + \overset{(v)}{\mathbb{S}}_{\text{deviatoric}} + \overset{(h)}{\mathbb{S}}_{\text{deviatoric}} \right)}^{\mathbb{S}_{\text{deviatoric}}} \\ &= \left( 3K + 3\tilde{K} + 3\zeta \frac{\partial}{\partial t} \right) \left( \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) + \left( 2\mu + 2\tilde{\mu} + 2\eta \frac{\partial}{\partial t} \right) \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) \end{aligned} \quad (164a)$$

$$= 3\hat{K} \left( \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) + 2\hat{\mu} \left( \mathbb{U} - \frac{1}{3} \mathbb{I} \text{Sp } \mathbb{U} \right) , \quad (164b)$$

where

$$\hat{K} \equiv K + \tilde{K} + \zeta \frac{\partial}{\partial t} \quad \text{and} \quad \hat{\mu} \equiv \mu + \tilde{\mu} + \eta \frac{\partial}{\partial t} . \quad (165)$$

As expected, a total linear deformation of an isotropic material can be described with two integro-differential operators, one acting on the volumetric strain, another on the deviatoric strain.

If an isotropic medium is also incompressible, the relative change of the volume vanishes:  $\text{Sp } \mathbb{U} = 0$ . Accordingly, the volumetric part of the strain becomes nil, and so do the volumetric parts of the elastic, hereditary, and viscous stresses. For such media, we end up with a simple relation which includes only deviators:

$$\overset{(total)}{\mathbb{S}} = \mathbb{S}_{\text{deviatoric}} = \overset{(e)}{\mathbb{S}}_{\text{deviatoric}} + \overset{(h)}{\mathbb{S}}_{\text{deviatoric}} + \overset{(v)}{\mathbb{S}}_{\text{deviatoric}} = 2\mu \mathbb{U} + 2\tilde{\mu} \mathbb{U} + 2\eta \frac{\partial}{\partial t} \mathbb{U} \quad (166)$$

or simply:

$$\overset{(total)}{\mathbb{S}} = \mathbb{S}_{\text{deviatoric}} = 2\hat{\mu} \mathbb{U} , \quad (167)$$

where  $\mathbb{U}$  contains only a deviatoric part, while

$$\hat{\mu} \equiv \mu + \tilde{\mu} + \eta \frac{\partial}{\partial t} \quad (168)$$

is the total, integro-differential operator, which is mapping the preceding history and the present rate of change of the strain to the present value of the stress.

It should be reiterated that the above approach is based on the assertion that the elastic, viscous, and hereditary stresses sum up, and that all three are related to the same total strain. A simple example of this approach, called the Kelvin-Voigt model, is rendered below in subsection A.6.3.

A different approach would be to assume that the strain consists of three distinct parts – elastic, hereditary, and viscous – and that these components are related to the same overall stress. A simple example of this treatment, termed the Maxwell model, is set out in subsection A.6.4. A more complex example of this approach is furnished by the Andrade model presented in subsection 5.3. Another way of combining elasticity and viscosity (with no hereditary reaction involved) is implemented by the Hohenemser-Prager (SAS) model explained in subsection A.6.5 below.

## A.6 Examples of viscoelastic behaviour with no hereditary reaction

### A.6.1 Elastic deformation

The truly simplest example of deformation is elastic:

$$\overset{(e)}{\mathbb{S}} = 2\mu \mathbb{U} \quad , \quad \mathbb{U} = J \overset{(e)}{\mathbb{S}} \quad , \quad (169)$$

where  $\mu$  and  $J$  are the unrelaxed rigidity and compliance:

$$\mu = \mu(0) \quad , \quad J = J(0) \quad , \quad \mu = 1/J \quad . \quad (170)$$

In the frequency domain, this relation assumes the same form as it would in the time domain:

$$\bar{\sigma}_{\gamma\nu}(\chi) = 2\mu \bar{u}_{\gamma\nu}(\chi) \quad , \quad 2\bar{u}_{\gamma\nu}(\chi) = J \bar{\sigma}_{\gamma\nu}(\chi) \quad . \quad (171)$$

### A.6.2 Viscous deformation

The next example is that of a purely viscous behaviour:

$$\overset{(v)}{\mathbb{S}} = 2\eta \frac{\partial}{\partial t} \mathbb{U} \quad . \quad (172)$$

It is straightforward from (172) and (27) that, in this regime, the Fourier components of the stress<sup>26</sup> are connected to those of the strain through

$$\bar{\sigma}_{\gamma\nu}(\chi) = 2\bar{\mu}(\chi) \bar{u}_{\gamma\nu}(\chi) \quad , \quad 2\bar{u}_{\gamma\nu}(\chi) = \bar{J}(\chi) \bar{\sigma}_{\gamma\nu}(\chi) \quad , \quad (173)$$

where the complex rigidity and the complex compliance are given by

$$\bar{\mu} = i\eta\chi \quad , \quad \bar{J} = -\frac{i}{\eta\chi} \quad . \quad (174)$$

---

<sup>26</sup> Although we no longer spell it out, the word *stress* everywhere means: *deviatoric stress*, as we agreed to consider the medium incompressible.

### A.6.3 Viscoelastic deformation: a Kelvin-Voigt material

The Kelvin-Voigt model, also called the Voigt model, can be represented with a purely viscous damper and a purely elastic spring connected in parallel. Subject to the same elongation, these elements have their forces summed up. This illustrates the situation where the total, viscoelastic stress consists of a purely viscous and a purely elastic inputs called into being by the same strain:

$$\mathbb{S} = \overset{(ve)}{\mathbb{S}} = \overset{(v)}{\mathbb{S}} + \overset{(e)}{\mathbb{S}} \quad , \quad \text{while} \quad \mathbb{U} = \overset{(v)}{\mathbb{U}} = \overset{(e)}{\mathbb{U}} \quad . \quad (175)$$

Then the total stress reads:

$$\mathbb{S} = \left( 2\mu + 2\eta \frac{\partial}{\partial t} \right) \mathbb{U} \quad , \quad (176a)$$

which is often presented in the form of

$$\mathbb{S} = 2\mu \left( \mathbb{U} + \tau_v \dot{\mathbb{U}} \right) \quad , \quad (176b)$$

with the so-called *Voigt time* defined as

$$\tau_v \equiv \eta/\mu \quad . \quad (177)$$

Comparing (176) with (46), we understand that the kernel of the rigidity operator for the Kelvin-Voigt model can be written down as

$$\mu(t - t') = \mu + \eta \delta(t - t') \quad . \quad (178)$$

Suppose the strain is varying in time as

$$u_{\gamma\nu}(t) = \frac{\sigma_0}{2\mu} \left[ 1 - \exp\left(-\frac{t - t_0}{\tau_v}\right) \right] \Theta(t - t_0) \quad , \quad (179)$$

so that

$$\dot{u}_{\gamma\nu}(t) = \frac{\sigma_0}{2\mu} \frac{1}{\tau_v} \exp\left(-\frac{t - t_0}{\tau_v}\right) \Theta(t - t_0) \quad . \quad (180)$$

Then insertion of (178) and (180) into (??) or, equivalently, insertion of (179) into (176a) demonstrates that this strain results from a stress<sup>27</sup>

$$\sigma_{\gamma\nu}(t) = \sigma_0 \Theta(t - t_0) \quad . \quad (181)$$

It would however be a mistake to deduce from this that the compliance function is equal to  $\mu^{-1} \left[ 1 - \exp\left(-\frac{t - t'}{\tau_v}\right) \right]$ , even though such a misstatement is sometimes made in the literature. This expression furnishes the compliance function only in the special situation of a Heaviside-step stress (181), but not in the general case.

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<sup>27</sup> For example, plugging of (178) and (180) into (??) leads to:

$$\begin{aligned} \sigma(t) &= \int_{t'=-\infty}^{t'=t} [\mu + \eta \delta(t - t')] \frac{\sigma_0}{\mu} \frac{1}{\tau_v} \exp\left(-\frac{t' - t_0}{\tau_v}\right) \Theta(t' - t_0) dt' \\ &= \begin{cases} \sigma_0 \left[ \int_{t'=t_0}^{t'=t} \exp\left(-\frac{t' - t_0}{\tau_v}\right) \frac{dt'}{\tau_v} + \exp\left(-\frac{t - t_0}{\tau_v}\right) \right] & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases} \quad , \end{aligned}$$

which is simply  $\sigma_0 \Theta(t - t_0)$ .

As can be easily shown from (27), in the frequency domain model (176) reads as (173), except that the complex rigidity and the complex compliance are now given by

$$\bar{\mu} = \mu (1 + i \chi \tau_v) \quad , \quad \bar{J} = \frac{J}{1 + i \chi \tau_v} \quad . \quad (182)$$

Recall that, for brevity, here and everywhere we write  $\mu$  and  $J$  instead of  $\mu(0)$  and  $J(0)$ .

The Kelvin-Voigt material becomes elastic in the low-frequency limit, and viscous in the high-frequency limit.

#### A.6.4 Viscoelastic deformation: a Maxwell material

The Maxwell model can be represented with a viscous damper and an elastic spring connected in series. Experiencing the same force, these elements have their elongations summed up. This example illustrates the situation where the total, viscoelastic strain consists of a purely viscous and a purely elastic contributions generated by the same stress  $\mathbb{S}$ :

$$\mathbb{U} = \overset{(v)}{\mathbb{U}} + \overset{(e)}{\mathbb{U}} \quad , \quad \text{where} \quad \overset{(e)}{\mathbb{S}} = 2\mu \overset{(e)}{\mathbb{U}} \quad \text{and} \quad \overset{(v)}{\mathbb{S}} = 2\eta \frac{\partial}{\partial t} \overset{(v)}{\mathbb{U}} \quad . \quad (183)$$

Since in the Maxwell regime both contributions to the strain are generated by the same stress

$$\mathbb{S} = \overset{(ve)}{\mathbb{S}} = \overset{(v)}{\mathbb{S}} = \overset{(e)}{\mathbb{S}} \quad , \quad (184)$$

formula (183) can be written down as

$$\dot{\mathbb{U}} = \frac{1}{2\mu} \dot{\mathbb{S}} + \frac{1}{2\eta} \mathbb{S} \quad (185a)$$

or, in a more conventional form:

$$\dot{\mathbb{S}} + \frac{1}{\tau_M} \mathbb{S} = 2\mu \dot{\mathbb{U}} \quad , \quad (185b)$$

with the so-called *Maxwell time* introduced as

$$\tau_M \equiv \eta/\mu \quad . \quad (186)$$

Although formally the Maxwell time is given by an expression mimicking the definition of the Voigt time, the meaning of these times is different.

Comparing (185) with the general expression (43) for the compliance operator, we see that, for the Maxwell model, the compliance operator in the time domain assumes the form:

$$J(t - t') = \left[ J + (t - t') \frac{1}{\eta} \right] \Theta(t - t') \quad , \quad (187)$$

where  $J \equiv 1/\mu$ . In the frequency domain, (185) can be written down as (173), with the complex rigidity and compliance given by

$$\bar{\mu}(\chi) = \mu \frac{i \chi \tau_M}{1 + i \chi \tau_M} \quad , \quad \bar{J}(\chi) = J \left( 1 - \frac{i}{\chi \tau_M} \right) = J - \frac{i}{\chi \eta} \quad . \quad (188)$$

Clearly, such a body becomes elastic in the high-frequency limit, and becomes viscous at low frequencies (the latter circumstance making the Maxwell model attractive to seismologists).



### A.6.5 Viscoelastic deformation: the Hohenemser-Prager (SAS) model

An attempt to combine the Kelvin-Voigt and Maxwell models leads to the Hohenemser-Prager model, also known as the Standard Anelastic Solid (SAS):

$$\tau_M \dot{\mathbb{S}} + \mathbb{S} = 2\mu \left( \mathbb{U} + \tau_V \dot{\mathbb{U}} \right) , \quad (189)$$

In the limit of  $\tau_M \rightarrow 0$ , this model approaches the one of Kelvin and Voigt (and  $\tau_V$  acquires the meaning of the Voigt time).

A transition from the SAS to Maxwell model, however, can be achieved only through re-definition of parameters. One should set:  $2\mu \rightarrow 0$  and  $\tau_V \rightarrow \infty$ , along with  $2\mu\tau_V \rightarrow 2\eta$ . Then (189) will become (185), with  $\tau_M$  playing the role of the Maxwell time.

In the frequency domain, (189) can be put into the form of (173), the complex rigidity and the complex compliance being expressed through the parameters as

$$\bar{\mu} = \mu \frac{1 + i\tau_V \chi}{1 + i\tau_M \chi} , \quad \bar{J} = J \frac{1 + i\tau_M \chi}{1 + i\tau_V \chi} . \quad (190)$$

This entails:  $\tan \delta \equiv \mathcal{I}m[\bar{\mu}]/\mathcal{R}e[\bar{\mu}] = \frac{(\tau_V - \tau_M)\chi}{1 + \tau_V \tau_M \chi^2}$ , whence it is easy to show that the tangent is related to its maximal value through

$$\tan \delta = 2 [\tan \delta]_{max} \frac{\tau \chi}{1 + \tau^2 \chi^2} , \quad \text{where } \tau \equiv \sqrt{\tau_M \tau_V} .$$

This is the so-called Debye peak, which is indeed observed in some materials.

To prove that the SAS solid is indeed anelastic, one has to make sure that a Heaviside step stress  $\Theta(t')$  entails a strain proportional to  $1 - \exp(-\Gamma t)$ , and to demonstrate that a predeformed sample subject to stress  $\Theta(-t')$  regains its shape as  $\exp(-\Gamma t)$ , where the relaxation constant  $\Gamma$  is positive. In subsection C.3 we shall do this for a SAS sphere.

## B Interconnection between the quality factor and the phase lag

The power  $P$  exerted by a tide-raising secondary on its primary can be written as

$$P = - \int \rho \vec{V} \cdot \nabla W d^3x \quad (191)$$

$\rho$ ,  $\vec{V}$ , and  $W$  signifying the density, velocity, and tidal potential in the small volume  $d^3x$  of the primary. The mass-conservation law  $\nabla \cdot (\rho \vec{V}) + \frac{\partial \rho}{\partial t} = 0$  enables one to shape the dot-product into the form of

$$\rho \vec{V} \cdot \nabla W = \nabla \cdot (\rho \vec{V} W) - \rho W \nabla \cdot \vec{V} - \vec{V} W \nabla \rho . \quad (192)$$

Under the realistic assumption of the primary's incompressibility, the term with  $\nabla \cdot \vec{V}$  may be omitted. To get rid of the term with  $\nabla \rho$ , one has to accept a much stronger approximation of the primary being homogeneous. Then the power will be rendered by

$$P = - \int \nabla \cdot (\rho \vec{V} W) d^3x = - \int \rho W \vec{V} \cdot \vec{n} dS , \quad (193)$$

$\vec{n}$  being the outward normal and  $dS$  being an element of the surface area of the primary. This expression for the power (pioneered, probably, by Goldreich 1963) enables one to calculate the work through radial displacements only, in neglect of horizontal motion.

Denoting the radial elevation with  $\zeta$ , we can write the power per unit mass,  $\mathcal{P} \equiv P/M$ , as:

$$\mathcal{P} = \left( -\frac{\partial W}{\partial r} \right) \vec{\mathbf{v}} \cdot \vec{\mathbf{n}} = \left( -\frac{\partial W}{\partial r} \right) \frac{d\zeta}{dt} . \quad (194)$$

A harmonic external potential

$$W = W_0 \cos(\omega_{lmpq} t) , \quad (195)$$

applied at a point of the primary's surface, will elevate this point by

$$\zeta = h_2 \frac{W_o}{g} \cos(\omega_{lmpq} t - \epsilon_{lmpq}) = h_2 \frac{W_o}{g} \cos(\omega_{lmpq} t - \omega_{lmpq} \Delta t_{lmpq}) , \quad (196)$$

with  $g$  being the surface gravity acceleration, and  $h_2$  denoting the Love number.

In formula (196),  $\omega_{lmpq}$  is one of the modes (105) showing up in the Darwin-Kaula expansion (103). The quantity  $\epsilon_{lmpq} = \omega_{lmpq} \Delta t_{lmpq}$  is the corresponding phase lag, while  $\Delta t_{lmpq}$  is the positively defined time lag at this mode. Although the tidal modes  $\omega_{lmpq}$  can assume any sign, both the potential  $W$  and elevation  $\zeta$  can be expressed via the positively defined forcing frequency  $\chi_{lmpq} = |\omega_{lmpq}|$  and the absolute value of the phase lag:

$$W = W_0 \cos(\chi t) , \quad (197)$$

$$\zeta = h_2 \frac{W_o}{g} \cos(\chi t - |\epsilon|) , \quad (198)$$

subscripts  $lmpq$  being dropped here and hereafter for brevity.

The vertical velocity of the considered element of the primary's surface will be

$$\frac{d\zeta}{dt} = -h_2 \chi \frac{W_o}{g} \sin(\chi t - |\epsilon|) = -h_2 \chi \frac{W_o}{g} (\sin \chi t \cos |\epsilon| - \cos \chi t \sin |\epsilon|) . \quad (199)$$

Introducing the notation  $A = h_2 \frac{W_o}{g} \frac{\partial W_0}{\partial r}$ , we write the power per unit mass as

$$\mathcal{P} = \left( -\frac{\partial W}{\partial r} \right) \frac{d\zeta}{dt} = A \chi \cos(\chi t) \sin(\chi t - |\epsilon|) , \quad (200)$$

and write the work  $w$  per unit mass, performed over a time interval  $(t_0, t)$ , as:

$$\begin{aligned} w|_{t_0}^t &= \int_{t_0}^t \mathcal{P} dt = A \int_{\chi t_0}^{\chi t} \cos(\chi t) \sin(\chi t - |\epsilon|) d(\chi t) = A \cos |\epsilon| \int_{\chi t_0}^{\chi t} \cos z \sin z dz - A \sin |\epsilon| \int_{\chi t_0}^{\chi t} \cos^2 z dz \\ &= -\frac{A}{4} [ \cos(2\chi t - |\epsilon|) + 2 \chi t \sin |\epsilon| ]_{t_0}^t . \end{aligned} \quad (201)$$

Being cyclic, the first term in (201) renders the elastic energy stored in the body. The second term, being linear in time, furnishes the energy damped. This clear interpretation of the two terms was offered by Stan Peale [2011, personal communication].

The work over a time period  $T = 2\pi/\chi$  is equal to the energy dissipated over the period:

$$w|_{t=0}^{t=T} = \Delta E_{cycle} = -A \pi \sin |\epsilon| . \quad (202)$$

It can be shown that the peak *work* is obtained over the time span from  $\pi$  to  $|\epsilon|$  and assumes the value

$$E_{peak}^{(work)} = \frac{A}{2} \left[ \cos |\epsilon| - \sin |\epsilon| \left( \frac{\pi}{2} - |\epsilon| \right) \right] , \quad (203)$$

whence the appropriate quality factor is given by:

$$Q_{work}^{-1} = \frac{-\Delta E_{cycle}}{2\pi E_{peak}^{(work)}} = \frac{\tan|\epsilon|}{1 - \left(\frac{\pi}{2} - |\epsilon|\right) \tan|\epsilon|} . \quad (204)$$

To calculate the peak *energy* stored in the body, we would note that the first term in (201) is maximal when taken over the span from  $\chi t = \pi/4 + |\epsilon|/2$  through  $\chi t = 3\pi/4 + |\epsilon|/2$ :

$$E_{peak}^{(energy)} = \frac{A}{2} , \quad (205)$$

and the corresponding quality factor is:

$$Q_{energy}^{-1} = \frac{-\Delta E_{cycle}}{2\pi E_{peak}^{(energy)}} = \sin|\epsilon| . \quad (206)$$

Goldreich (1963) suggested to employ the span  $\chi t = (0, \pi/4)$ . The absolute value of the resulting power, denoted in *Ibid.* as  $E^*$ , is equal to

$$E^* = \frac{A}{2} \cos|\epsilon| \quad (207)$$

and is *not* the peak value of the energy stored nor of the work performed. Goldreich (1963) however employed it to define a quality factor, which we shall term  $Q_{Goldreich}$ . This factor is introduced via

$$Q_{Goldreich}^{-1} = \frac{-\Delta E_{cycle}}{2\pi E^*} = \tan|\epsilon| . \quad (208)$$

In our opinion, the quality factor  $Q_{energy}$  defined through (206) is preferable, because the expansion of tides contains terms proportional to  $k_l(\chi_{lmpq}) \sin \epsilon_l(\chi_{lmpq})$ . Since the long-established tradition suggests to substitute  $\sin \epsilon$  with  $1/Q$ , it is advisable to define the  $Q$  exactly as (206), and also to call it  $Q_l$ , to distinguish it from the seismic quality factor (Efroimsky 2012).

## C Tidal response of a homogeneous viscoelastic sphere (Churkin 1998)

This section presents some results from the unpublished preprint by Churkin (1998). We took the liberty of upgrading the notations<sup>28</sup> and correcting some minor oversights.

### C.1 A homogeneous Kelvin-Voigt spherical body

In combination with the Correspondence Principle, the formulae from subsection A.6.3 furnish the following expression for the complex Love numbers of a Kelvin-Voigt body:

$$\bar{k}_l(\chi) = \frac{3}{2(l-1)} \frac{1}{1 + A_l (1 + \tau_V i \chi)} , \quad (209)$$

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<sup>28</sup> Churkin (1998) employed the notation  $k_l(\tau)$  for what we call  $\dot{k}_l(\tau)$ . Our notations are more convenient in that they amplify the close analogy between the Love functions and the compliance function.

It then can be demonstrated, with aid of (61), that the time-derivative of the corresponding Love function is

$$\dot{k}_l(\tau) = \begin{cases} \frac{3}{2(l-1)} \frac{1}{A_l \tau_V} \exp(-\tau \zeta_l) \Theta(\tau) & \text{for } \tau_V > 0 \\ \frac{3}{2(l-1)} \frac{1}{1 + A_l} \delta(\tau) & \text{for } \tau_V = 0 \end{cases} , \quad (210)$$

while the Love function itself has the form of

$$\begin{aligned} k_l(\tau) &= \frac{3}{2(l-1)} \frac{1}{A_l \zeta_l \tau_V} \left[ 1 - \exp(-\tau \zeta_l) \right] \Theta(\tau) \\ &= \frac{3}{2(l-1)} \frac{1}{1 + A_l} \left[ 1 - \exp(-\tau \zeta_l) \right] \Theta(\tau) , \end{aligned} \quad (211)$$

where

$$\zeta_l \equiv \frac{1 + A_l}{A_l \tau_V} . \quad (212)$$

Formulae (210) may look confusing, in that  $\exp(-\tau \zeta_l)$  simply vanishes in the elastic limit, i.e., when  $\tau_V \rightarrow 0$  and  $\zeta_l \rightarrow \infty$ . We however should not be misled by this mathematical artefact stemming from the nonanalyticity of the exponent function. Instead, we should keep in mind that a physical meaning is attributed not to the Love functions or their derivatives but to the results of the Love operator's action on realistic disturbances. For example, a Heaviside step potential

$$W_l(\vec{R}, \vec{r}^*, t') = W \Theta(t') \quad (213)$$

applied to a homogeneous Kelvin-Voigt spherical body will furnish, through relation (60), the following response of the potential:

$$\begin{aligned} U_l(\vec{r}, t) &= \left( \frac{R}{r} \right)^{l+1} \int_{t'=-\infty}^{t'=t} \dot{k}_l(t-t') W \Theta(t') dt' = \left( \frac{R}{r} \right)^{l+1} \int_{t'=0}^{t'=t} \dot{k}_l(t-t') W dt' \\ &= W \left( \frac{R}{r} \right)^{l+1} \int_{\tau=0}^{\tau=t} \dot{k}_l(\tau) d\tau = \frac{3}{2(l-1)} \frac{1 - \exp(-t \zeta_l)}{1 + A_l} \left( \frac{R}{r} \right)^{l+1} W . \end{aligned} \quad (214)$$

In the elastic limit, this becomes:

$$\tau_V \rightarrow 0 \implies \zeta_l \rightarrow \infty \implies U_l(\vec{r}, t) \rightarrow \frac{3}{2(l-1)} \frac{1}{1 + A_l} \left( \frac{R}{r} \right)^{l+1} W , \quad (215)$$

which reproduces the case described by the static Love number  $k_l = \frac{3}{2} \frac{1}{1 + A_l}$ .

An alternative way of getting (215) would be to employ formulae (211) and (59a).

## C.2 A homogeneous Maxwell spherical body

Using the formulae presented in the Appendix A.6.4, and relying upon the Correspondence Principle, we write down the complex Love numbers for a Maxwell material as

$$\bar{k}_l(\chi) = \frac{3}{2(l-1)} \frac{1}{1 + \frac{A_l \tau_M i \chi}{1 + \tau_M i \chi}} = \frac{3}{2(l-1)} \frac{1}{1 + A_l} \left[ 1 + \frac{A_l}{1 + (1 + A_l) \tau_M i \chi} \right] , \quad (216)$$

which corresponds, via (61), to

$$\dot{k}_l(\tau) = \frac{3}{2(l-1)} \frac{\delta(\tau) + A_l \gamma_l \exp(-\tau \gamma_l) \Theta(\tau)}{1 + A_l} \quad (217)$$

and

$$k_l(\tau) = \frac{3}{2(l-1)} \frac{1 + A_l \left[ 1 - \exp(-\tau \gamma_l) \right]}{1 + A_l} \Theta(\tau) \quad , \quad (218)$$

where

$$\gamma_l \equiv \frac{1}{(1 + A_l) \tau_M} \quad . \quad (219)$$

A Heaviside step potential

$$W_l(\vec{R}, \vec{r}^*, t') = W \Theta(t') \quad (220)$$

will, according to formula (60), render the following response:

$$\begin{aligned} U_l(\vec{r}, t) &= \left( \frac{R}{r} \right)^{l+1} \int_{t'=-\infty}^{t'=t} \dot{k}_l(t-t') W \Theta(t') dt' = \left( \frac{R}{r} \right)^{l+1} \int_{t'=0}^{t'=t} \dot{k}_l(t-t') W dt' \\ &= W \left( \frac{R}{r} \right)^{l+1} \int_{\tau=0}^{\tau=t} \dot{k}_l(\tau) d\tau = \frac{3}{2(l-1)} \frac{1 + A_l \left[ 1 - \exp(-t \gamma_l) \right]}{1 + A_l} \left( \frac{R}{r} \right)^{l+1} W \Theta(t) . \end{aligned} \quad (221)$$

In the elastic limit, we obtain:

$$\tau_M \rightarrow \infty \implies \gamma_l \rightarrow 0 \implies U_l(\vec{r}, t) \rightarrow \frac{3}{2(l-1)} \frac{1}{1 + A_l} \left( \frac{R}{r} \right)^{l+1} W \quad , \quad (222)$$

which corresponds to the situation described by the static Love number  $k_l = \frac{3}{2(l-1)} \frac{1}{1 + A_l}$ .

### C.3 A homogeneous Hohenemser-Prager (SAS) spherical body

The Correspondence Principle, along with the formulae from subsection A.6.5, yields the following expression for the complex Love numbers of a Hohenemser-Prager (SAS) spherical body:

$$\bar{k}_l(\chi) = \frac{3}{2(l-1)} \frac{1}{1 + A_l \frac{1 + i\chi \tau_V}{1 + i\chi \tau_M}} \quad . \quad (223)$$

Combined with (61), this entails:

$$\dot{k}_l(\tau) = \frac{3}{2(l-1)} \frac{1}{1 + A_l \frac{\tau_V}{\tau_M}} \left[ \delta(\tau) + \frac{A_l}{\tau_M} \frac{\tau_V - \tau_M}{\tau_M + A_l \tau_V} \exp\left(-\frac{1 + A_l}{\tau_M + A_l \tau_V} \tau\right) \right] \quad (224)$$

and

$$k_l(\tau) = \frac{3}{2(l-1)} \frac{1 - \frac{A_l}{\tau_M} \frac{\tau_V - \tau_M}{1 + A_l} \left[ 1 - \exp\left(-\frac{1 + A_l}{\tau_M + A_l \tau_V} \tau\right) \right]}{1 + A_l \frac{\tau_V}{\tau_M}} \Theta(\tau) \quad (225)$$

A Heaviside step potential

$$W_l(\vec{R}, \vec{r}^*, t') = W \Theta(t') \quad (226)$$

applied to a SAS spherical body will then result in the following variation of its potential:

$$U_l(\vec{r}, t) =$$

$$\begin{aligned} \left(\frac{R}{r}\right)^{l+1} \int_{t'=-\infty}^{t'=t} \dot{k}_l(t-t') W \Theta(t') dt' &= \left(\frac{R}{r}\right)^{l+1} \int_{t'=0}^{t'=t} \dot{k}_l(t-t') W dt' = W \left(\frac{R}{r}\right)^{l+1} \int_{\tau=0}^{\tau=t} \dot{k}_l(\tau) d\tau \\ &= \frac{3}{2(l-1)} \left( \frac{1}{1+A_l \frac{\tau_V}{\tau_M}} + \frac{A_l}{1+A_l} \frac{\tau_V - \tau_M}{\tau_M + A_l \tau_V} \right) \left[ 1 - \exp\left(-\frac{1+A_l}{\tau_M + A_l \tau_V} t\right) \right] \left(\frac{R}{r}\right)^{l+1} W \Theta(t) . \end{aligned} \quad (227)$$

Within this model, the elastic limit is achieved by setting  $\tau_M = \tau_V$ , whence we obtain the case described by the static Love number  $k_l = \frac{3}{2} \frac{1}{1+A_l}$ . Interestingly, the elastic regime is achieved even when these times are not zero. Their being equal to one another turns out to be sufficient.

Repeating the above calculation for tidal disturbance  $W \Theta(-t')$ , we shall see that, after the tidal perturbation is removed, a tidally prestressed sphere regains its shape, the stress relaxing at a rate proportional to  $\exp\left(-\frac{1+A_l}{\tau_M + A_l \tau_V} t\right)$ .

## D The correspondence principle (elastic-viscoelastic analogy)

### D.1 The correspondence principle, for nonrotating bodies

While the static Love numbers depend on the static rigidity  $\mu$  through (3), it is not immediately clear if a similar formula interconnects also  $\bar{k}_l(\chi)$  with  $\bar{\mu}(\chi)$ . To understand why and when the relation should hold, recall that formulae (3) originate from the solution of a boundary-value problem for a system incorporating two equations:

$$\sigma_{\beta\nu} = 2\mu u_{\beta\nu} , \quad (228a)$$

$$0 = \frac{\partial \sigma_{\beta\nu}}{\partial x_\nu} - \frac{\partial p}{\partial x_\beta} - \rho \frac{\partial(W+U)}{\partial x_\beta} , \quad (228b)$$

the latter being simply the equation of equilibrium written for a *static* viscoelastic medium, in neglect of compressibility and heat conductivity. The notations  $\sigma_{\beta\nu}$  and  $u_{\beta\nu}$  stand for the *deviatoric* stress and strain,  $p \equiv -\frac{1}{3} \text{Sp} \mathbb{S}$  is the pressure (set to be nil in incompressible media), while  $W$  and  $U$  are the perturbing and perturbed potentials. By solving the system, one arrives at the static relation  $U_l = k_l W_l$ , with the customary static Love numbers  $k_l$  expressed via  $\rho$ ,  $R$ , and  $\mu$  by (3).

Now let us write equation like (228a - 228b) for time-dependent deformation of a *nonrotating* body:

$$\mathbb{S} = 2\hat{\mu} \mathbb{U} , \quad (229a)$$

$$\rho \ddot{\mathbf{u}} = \nabla \mathbb{S} - \nabla p - \nabla(W+U) \quad (229b)$$

or, in terms of components:

$$\sigma_{\beta\nu} = 2 \hat{\mu} u_{\beta\nu} , \quad (230a)$$

$$\rho \ddot{u}_\beta = \frac{\partial \sigma_{\beta\nu}}{\partial x_\nu} - \frac{\partial p}{\partial x_\beta} - \rho \frac{\partial(W + U)}{\partial x_\beta} . \quad (230b)$$

In the frequency domain, this will look:

$$\bar{\sigma}_{\beta\nu}(\chi) = 2 \bar{\mu}(\chi) \bar{u}_{\beta\nu}(\chi) , \quad (231a)$$

$$\rho \chi^2 \bar{u}_{\beta\nu}(\chi) = \frac{\partial \bar{\sigma}_{\beta\nu}(\chi)}{\partial x_\nu} - \frac{\partial \bar{p}(\chi)}{\partial x_\beta} - \rho \frac{\partial [\bar{W}(\chi) + \bar{U}(\chi)]}{\partial x_\beta} , \quad (231b)$$

where a bar denotes a spectral component for all functions except  $\mu$  – recall that  $\bar{\mu}$  is a spectral component not of the kernel  $\mu(\tau)$  but of its time-derivative  $\dot{\mu}(\tau)$ .

Unless the frequencies are extremely high, we can neglect the body-fixed acceleration term  $\chi^2 \bar{u}_{\beta\nu}(\chi)$  in the second equation, in which case our system of equations for the spectral components will mimic (228). Thus we arrive at the so-called *correspondence principle* (also known as the *elastic-viscoelastic analogy*), which maps a solution of a linear viscoelastic boundary-value problem to a solution of a corresponding elastic problem with the same initial and boundary conditions. As a result, the algebraic equations for the Fourier (or Laplace) components of the strain and stress in the viscoelastic case mimic the equations connecting the strain and stress in the appropriate elastic problem. So the viscoelastic operational moduli  $\bar{\mu}(\chi)$  or  $\bar{J}(\chi)$  obey the same algebraic relations as the elastic parameters  $\mu$  or  $J$ .

In the literature, there is no consensus on the authorship of this principle. For example, Haddad (1995) mistakenly attributes it to several authors who published in the 1950s and 1960s. In reality, the principle was pioneered almost a century earlier by Darwin (1879), for isotropic incompressible media. The principle was extended to more general types of media by Biot (1954, 1958), who also pointed out some limitations of this principle.

## D.2 The correspondence principle, for rotating bodies

Consider a body of mass  $M_{prim}$ , which is spinning at a rate  $\vec{\omega}$  and is also performing some orbital motion (for example, is orbiting, with its partner of mass  $M_{sec}$ , around their mutual centre of mass). Relative to some inertial coordinate system, the centre of mass of the body is located at  $\vec{x}_{CM}$ , while a small parcel of its material is positioned at  $\vec{x}$ . Relative to the centre of mass of the body, the parcel is located at  $\vec{r} = \vec{x} - \vec{x}_{CM}$ . The body being deformable, we can decompose  $\vec{r}$  into its average value,  $\vec{r}_0$ , and an instantaneous displacement  $\vec{u}$ :

$$\left. \begin{aligned} \vec{x} &= \vec{x}_{CM} + \vec{r} \\ \vec{r} &= \vec{r}_0 + \vec{u} \end{aligned} \right\} \implies \vec{x} = \vec{x}_{CM} + \vec{r}_0 + \vec{u} . \quad (232)$$

Denote with  $D/Dt$  the time-derivative in the inertial frame. The symbol  $d/dt$  and its synonym, overdot, will be reserved for the time-derivative in the body frame, so  $d\vec{r}_0/dt = 0$ . Then

$$\frac{D\vec{r}}{Dt} = \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r} \quad \text{and} \quad \frac{D^2\vec{r}}{Dt^2} = \frac{d^2\vec{r}}{dt^2} + 2 \vec{\omega} \times \frac{d\vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} . \quad (233)$$

Together, the above formulae result in

$$\begin{aligned} \frac{D^2\vec{x}}{Dt^2} &= \frac{D^2\vec{x}_{CM}}{Dt^2} + \frac{D^2\vec{r}}{Dt^2} = \frac{D^2\vec{x}_{CM}}{Dt^2} + \frac{d^2\vec{r}}{dt^2} + 2 \vec{\omega} \times \frac{d\vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} \\ &= \frac{D^2\vec{x}_{CM}}{Dt^2} + \frac{d^2\vec{u}}{dt^2} + 2 \vec{\omega} \times \frac{d\vec{u}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} . \end{aligned} \quad (234)$$

The equation of motion for a small parcel of the body's material will read as

$$\rho \frac{D^2 \vec{x}}{Dt^2} = \nabla \mathbb{S} - \nabla p + \vec{F}_{self} + \vec{F}_{ext} \quad , \quad (235)$$

where  $\vec{F}_{ext}$  is the exterior gravity force *per unit volume*, while  $\vec{F}_{self}$  is the “interior” gravity force *per unit volume*, i.e., the self-force wherewith the rest of the body is acting upon the selected parcel of medium. Insertion of (234) in (235) furnishes:

$$\rho \left[ \frac{D^2 \vec{x}_{CM}}{Dt^2} + \ddot{\vec{u}} + 2\vec{\omega} \times \dot{\vec{u}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} \right] = \nabla \mathbb{S} - \nabla p + \vec{F}_{self} + \vec{F}_{ext} \quad . \quad (236)$$

At the same time, for the primary body as a whole, we can write:

$$M_{prim} \frac{D^2 \vec{x}_{CM}}{Dt^2} = \int_V \vec{F}_{ext} d^3 \vec{r} \quad , \quad (237)$$

the integration being carried out over the volume  $V$  of the primary. (Recall that  $\vec{F}_{ext}$  is a force per unit volume.) Combined together, the above two equations will result in

$$\rho \left[ \ddot{\vec{u}} + 2\vec{\omega} \times \dot{\vec{u}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} \right] = \nabla \mathbb{S} - \nabla p + \vec{F}_{self} + \vec{F}_{ext} - \frac{\rho}{M_{prim}} \int_V \vec{F}_{ext} d^3 \vec{r} \quad . \quad (238)$$

For a spherically-symmetrical (not necessarily radially-homogeneous) body, the integral on the right-hand side clearly removes the Newtonian part of the force, leaving the harmonics intact:

$$\vec{F}_{ext} - \frac{\rho}{M_{prim}} \int_V \vec{F}_{ext} d^3 \vec{r} = \rho \sum_{l=2}^{\infty} \nabla W_l \quad , \quad (239)$$

where the harmonics are given by

$$W_l(\vec{r}, \vec{r}^*) = - \frac{G M_{sec}}{r^*} \left( \frac{r}{r^*} \right)^l P_l(\cos \gamma) \quad , \quad (240)$$

$\vec{r}^*$  being the vector pointing from the centre of mass of the primary to that of the secondary, and  $\gamma$  being the angular separation between  $\vec{r}$  and  $\vec{r}^*$ , subtended at the centre of mass of the primary.

In reality, a tiny extra force  $\mathcal{F}$ , the tidal force per unit volume, is left over due to the body being slightly distorted:

$$\vec{F}_{ext} - \frac{\rho}{M_{prim}} \int_V \vec{F}_{ext} d^3 \vec{r} = \rho \sum_{l=2}^{\infty} \nabla W_l + \mathcal{F} \quad . \quad (241)$$

Here  $\mathcal{F}$  is the density multiplied by the average tidal acceleration experienced by the body as a whole. In neglect of  $\mathcal{F}$ , we arrive at

$$\rho \left[ \ddot{\vec{u}} + 2\vec{\omega} \times \dot{\vec{u}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} \right] = \nabla \mathbb{S} - \nabla p - \rho \sum_{l=2}^{\infty} \nabla (U_l + W_l) \quad . \quad (242)$$

Here, to each disturbing term of the exterior potential,  $W_l$ , corresponds a term  $U_l$  of the self-potential, the self-force thus being expanded into  $\vec{F}_{self} = - \sum_{l=2}^{\infty} \nabla U_l$ .

Equation (242) could as well have been derived in the body frame, where it would have assumed the same form.

Denoting the tidal frequency with  $\chi$ , we see that the terms on the left-hand side have the order of  $\rho \chi^2 u$ ,  $\rho \omega \chi u$ ,  $\rho \omega^2 r$ , and  $\rho \dot{\chi} \omega r$ , correspondingly. In realistic situations, the first two terms, thus, can be neglected, and we end up with

$$0 = \nabla \mathbb{S} - \nabla p - \rho \sum_{l=2}^{\infty} \nabla (U_l + W_l) - \rho \vec{\omega} \times (\vec{\omega} \times \vec{r}) - \rho \dot{\vec{\omega}} \times \vec{r} \quad , \quad (243)$$

the term  $-\nabla p$  vanishing in an incompressible media.



### D.3 The centripetal term and the zero-degree Love number

The centripetal term in (243) can be split into a purely radial part and a part that can be incorporated into the  $W_2$  term of the tide-raising potential, as was suggested by Love (1909, 1911). Introducing the colatitude  $\phi'$  through  $\cos \phi' = \frac{\vec{\omega}}{|\vec{\omega}|} \cdot \frac{\vec{r}}{|\vec{r}|}$ , we can write down the evident equality

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} (\vec{\omega} \cdot \vec{r}) - \vec{r} \vec{\omega}^2 = \nabla \left[ \frac{1}{2} (\vec{\omega} \cdot \vec{r})^2 - \frac{1}{2} \vec{\omega}^2 \vec{r}^2 \right] = \nabla \left[ \frac{1}{2} \vec{\omega}^2 \vec{r}^2 (\cos^2 \phi' - 1) \right] .$$

The definition  $P_2(\cos \phi') = \frac{1}{2} (3 \cos^2 \phi' - 1)$  easily renders:  $\cos^2 \phi' = \frac{2}{3} P_2(\cos \phi') + \frac{1}{3}$ , whence:

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \nabla \left[ \frac{1}{3} \vec{\omega}^2 \vec{r}^2 [P_2(\cos \phi') - 1] \right] . \quad (244)$$

We see that the centripetal force splits into a second-harmonic and purely-radial parts:

$$- \rho \vec{\omega} \times (\vec{\omega} \times \vec{r}) = - \nabla \left[ \frac{\rho}{3} \vec{\omega}^2 \vec{r}^2 P_2(\cos \phi') \right] + \nabla \left[ \frac{\rho}{3} \vec{\omega}^2 \vec{r}^2 \right] , \quad (245)$$

where we assume the body to be homogeneous. The second-harmonic part can be incorporated into the external potential. The response to this part will be proportional to the degree-2 Love number  $k_2$ .

The purely radial part of the centripetal potential generates a radial deformation. This part of the potential is often ignored, the associated deformation being tacitly included into the equilibrium shape of the body. Compared to the main terms of the equation of motion, this radial term is of the order of  $10^{-3}$  for the Earth, and is smaller for most other bodies. As the rotation variations of the Earth are of the order of  $10^{-5}$ , this term leads to a tiny change in the geopotential and to an associated displacement of the order of a micrometer.<sup>29</sup>

However, for other rotators the situation may be different. For example, in Phobos, whose libration magnitude is large (about 1 degree), the radial term may cause an equipotential-surface variation of about 10 cm. This magnitude is large enough to be observed by future missions and should be studied in more detail.<sup>30</sup> The emergence of the purely radial deformation gives birth to the zero-degree Love number (Dahlen 1976, Matsuyama & Bills 2010). Using Dahlen's results, Yoder (1982, eqns 21 - 22) demonstrated that the contribution of the radial part of the centripetal potential to the change in mean motion of Phobos is about 3%, which is smaller than the uncertainty in our knowledge of Phobos'  $k_2/Q$ . It should be mentioned, however, that the calculations by Dahlen (1976) and Matsuyama & Bills (2010) were performed for steady (or slowly changing) rotation, and not for libration. This means that Yoder's application of Dahlen's result to Phobos requires extra justification.

What is important for us here is that the radial term does not interfere with the calculation of the Love number. Being independent of the longitude, this term generates no tidal torque either, provided the obliquity is neglected.

### D.4 The toroidal term

The inertial term  $-\rho \dot{\vec{\omega}} \times \vec{r}$  in the equation of motion (243) can be cast into the form

$$-\rho \dot{\vec{\omega}} \times \vec{r} = \rho \vec{r} \times \nabla(\dot{\vec{\omega}} \cdot \vec{r}) , \quad (246)$$

<sup>29</sup> Tim Van Hoolst, private communication.

<sup>30</sup> Tim Van Hoolst, private communication.

whence we see that this term is of a toroidal type. Being almost nil for a despinning primary, this force becomes important for a librating object.

In spherically-symmetric bodies, the toroidal force (246) generates toroidal deformation only. This deformation produces neither radial uplifts nor variations of the gravitational potential. Hence its presence does not influence the expressions for the Love numbers associated with vertical displacement ( $h_l$ ) or the potential ( $k_l$ ). As this deformation yields no change in the gravitational potential of the tidally-perturbed body, there is no tidal torque associated with this deformation. Being divergence-free, this deformation entails no contraction or expansion either, i.e., it is purely shear. Still, this deformation contributes to dissipation. Besides, since the toroidal forcing results in the toroidal deformation, it can, in principle, be associated with a “toroidal” Love number.

To estimate the dissipation caused by the toroidal rotational force, Yoder (1982) introduced an equivalent effective torque. He pointed out that this force becomes important when the magnitude of the physical libration is comparable to that of the optical libration. According to *Ibid.*, the toroidal force contributes to the change of the mean motion of Phobos about 1.6%, which is less than the input from the purely radial part.

## E The Andrade and Maxwell models at different frequencies

### E.1 Response of a sample obeying the Andrade model

Within the Andrade model, the tangent of the phase lag demonstrates the so-called “elbow dependence”. At high frequencies, the tangent of the lag obeys a power law with an exponent equal to  $-\alpha$ , where  $0 < \alpha < 1$ . At low frequencies, the tangent of the lag once again obeys a power law, this time though with an exponent  $-(1 - \alpha)$ . This model fits well the behaviour of ices, metals, silicate rocks, and many other materials.

However the applicability of the Andrade law may depend upon the intensity of the load and, accordingly, upon the damping mechanisms involved. Situations are known, when, at low frequencies, anelasticity becomes much less efficient than viscosity. In these cases, the Andrade model approaches, at low frequencies, the Maxwell model.

#### E.1.1 The high-frequency band

At high frequencies, expression (89b) gets simplified. In the numerator, the term with  $z^{-\alpha}$  dominates:  $z^{-\alpha} \gg z^{-1} \zeta$ , which is equivalent to  $z \gg \zeta^{\frac{1}{1-\alpha}}$ . In the denominator, the constant term dominates:  $1 \gg z^{-\alpha}$ , or simply:  $z \gg 1$ . To know which of the two conditions,  $z \gg \zeta^{\frac{1}{1-\alpha}}$  or  $z \gg 1$ , is stronger, we recall that at high frequencies anelasticity beats viscosity. So the  $\alpha$ -term in (85) is large enough. In other words, the Andrade timescale  $\tau_A$  should be smaller (or, at least, not much higher) than the viscoelastic time  $\tau_M$ . Accordingly, at high frequencies,  $\zeta$  is smaller (or, at least, not much higher) than unity. Hence, within the high-frequency band, either the condition  $z \gg 1$  is stronger than  $z \gg \zeta^{\frac{1}{1-\alpha}}$  or the two conditions are about equivalent. This, along with (90) and (91) enables us to write:

$$\tan \delta \approx (\chi \tau_A)^{-\alpha} \sin \left( \frac{\alpha \pi}{2} \right) \Gamma(\alpha + 1) \quad \text{for } \chi \gg \tau_A^{-1} = \tau_M^{-1} \zeta^{-1} \quad . \quad (247)$$

The tangent being small, the expression for  $\sin \delta$  looks identical:

$$\sin \delta \approx (\chi \tau_A)^{-\alpha} \sin \left( \frac{\alpha \pi}{2} \right) \Gamma(\alpha + 1) \quad \text{for } \chi \gg \tau_A^{-1} = \tau_M^{-1} \zeta^{-1} \quad . \quad (248)$$

### E.1.2 The intermediate region

In the intermediate region, the behaviour of the phase lag  $\delta$  depends upon the frequency-dependence of  $\zeta$ . For example, if there happens to exist an interval of frequencies over which the conditions  $1 \gg z \gg \zeta^{\frac{1}{1-\alpha}}$  are obeyed, then over this interval we shall have:  $1 \ll z^{-\alpha}$  and  $z^{-\alpha} \gg \zeta z^{-1}$ . Applying these inequalities to (89b), we see that over such an interval of frequencies  $\tan \delta$  will behave as  $z^{-2\alpha} \tan\left(\frac{\alpha\pi}{2}\right)$ .

### E.1.3 The low-frequency band

At low frequencies, the term  $z^{-1}\zeta$  becomes leading in the numerator of (89b):  $z^{-\alpha} \ll z^{-1}\zeta$ , which requires  $z \ll \zeta^{\frac{1}{1-\alpha}}$ . In the denominator, the term with  $z^{-\alpha}$  becomes the largest:  $1 \ll z^{-\alpha}$ , whence  $z \ll 1$ . Since at low frequencies the viscous term in (85) is larger than the anelastic term, we expect that for these frequencies  $\zeta$  is larger (at least, not much smaller) than unity. Thence the condition  $z \ll 1$  becomes sufficient. Its fulfilment ensures the fulfilment of  $z \ll \zeta^{\frac{1}{1-\alpha}}$ . Thus we state:

$$\tan \delta \approx (\chi \tau_A)^{-(1-\alpha)} \frac{\zeta}{\cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha+1)} \quad \text{for} \quad \chi \ll \tau_A^{-1} = \tau_M^{-1} \zeta^{-1} \quad . \quad (249)$$

The appropriate expression for  $\sin \delta$  will be:

$$\sin \delta \approx 1 - O\left((\chi \tau_A)^{2(1-\alpha)} \zeta^{-2}\right) \quad \text{for} \quad \chi \ll \tau_A^{-1} = \tau_M^{-1} \zeta^{-1} \quad , \quad (250)$$

It would be important to emphasise that the threshold  $\tau_A^{-1} = \tau_M^{-1} \zeta^{-1}$  standing in (247) and (248) is *different* from the threshold  $\tau_A^{-1} = \tau_M^{-1} \zeta^{-1}$  showing up in (249) and (250), even though these two thresholds are given by the same expression. The reason for this is that the timescales  $\tau_A$  and  $\tau_M$  are not fixed constants. While the Maxwell time is likely to be a very slow function of the frequency, the Andrade time may undergo a faster change over the transitional region:  $\tau_A$  must be larger than  $\tau_M$  at low frequencies (so anelasticity yields to viscosity), and must become shorter than or of the order of  $\tau_M$  at high frequencies (so anelasticity becomes stronger). This way, the threshold  $\tau_A^{-1}$  standing in (249 - 250) is lower than the threshold  $\tau_A^{-1}$  standing in (247 - 248). The gap between these thresholds is the region intermediate between the two pronounced power laws (247) and (249).

### E.1.4 The low-frequency band: a special case, the Maxwell model

Suppose that, below some threshold  $\chi_0$ , anelasticity quickly becomes *much less* efficient than viscosity. This would imply a steep increase of  $\zeta$  (equivalently, of  $\tau_A$ ) at low frequencies. Then, in (89b), we shall have:  $1 \gg z^{-\alpha}$  and  $z^{-\alpha} \ll \zeta z^{-1}$ . This means that, for frequencies below  $\chi_0$ , the tangent will behave as

$$\tan \delta \approx z^{-1} \zeta = (\chi \tau_M)^{-1} \quad \text{for} \quad \chi \ll \chi_0 \quad . \quad (251)$$

the well-known viscous scaling law for the lag.

The study of ices and minerals under weak loads (Castillo-Rogez et al. 2011, Castillo-Rogez 2011) has not shown such an abrupt vanishing of anelasticity. However, Karato & Spetzler (1990) point out that this should be happening in the Earth's mantle, where the loads are much higher and anelasticity is caused by unpinning of dislocations.

## E.2 The behaviour of $|k_l(\chi)| \sin \epsilon_l(\chi) = -\mathcal{I}m [\bar{k}_l(\chi)]$ within the Andrade and Maxwell models

As we explained in subsection 4.1, products  $k_l(\chi_{lmpq}) \sin \epsilon_l(\chi_{lmpq})$  enter the  $lmpq$  term of the Darwin-Kaula series for the tidal potential, force, and torque. Hence the importance to know the behaviour of these products as functions of the tidal frequency  $\chi_{lmpq}$ .

### E.2.1 Prefatory algebra

It ensues from (68) that

$$\bar{k}_l(\chi) = \frac{3}{2(l-1)} \frac{(\mathcal{R}e [\bar{J}(\chi)])^2 + (\mathcal{I}m [\bar{J}(\chi)])^2 + A_l J \mathcal{R}e [\bar{J}(\chi)] + i A_l J \mathcal{I}m [\bar{J}(\chi)]}{(\mathcal{R}e [\bar{J}(\chi)] + A_l J)^2 + (\mathcal{I}m [\bar{J}(\chi)])^2}, \quad (252)$$

whence

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) = -\mathcal{I}m [\bar{k}_l(\chi)] = \frac{3}{2(l-1)} \frac{-A_l J \mathcal{I}m [\bar{J}(\chi)]}{(\mathcal{R}e [\bar{J}(\chi)] + A_l J)^2 + (\mathcal{I}m [\bar{J}(\chi)])^2}, \quad (253)$$

$J = J(0) \equiv 1/\mu = 1/\mu(0)$  being the unrelaxed compliance (the inverse of the unrelaxed shear modulus  $\mu$ ). For an Andrade material, the compliance  $\bar{J}$  in the frequency domain is rendered by (86). Its imaginary and real parts are given by (87 - 88). It is then easier to rewrite (253) as

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) =$$

$$\frac{3 A_l}{2(l-1)} \frac{\zeta z^{-1} + z^{-\alpha} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha+1)}{\left[A_l + 1 + z^{-\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha+1)\right]^2 + \left[\zeta z^{-1} + z^{-\alpha} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(1+\alpha)\right]^2}, \quad (254)$$

where

$$z \equiv \chi \tau_A = \chi \tau_M \zeta \quad (255)$$

and

$$\zeta \equiv \frac{\tau_A}{\tau_M}. \quad (256)$$

For  $\beta \rightarrow 0$ , i.e., for  $\tau_A \rightarrow \infty$ , (254) coincides with the appropriate expression for a spherical Maxwell body.

### E.2.2 The high-frequency band

Within the upper band, the term with  $z^{-\alpha}$  dominates the numerator, while  $A_l$  dominates the denominator. The domination of  $z^{-\alpha}$  in the numerator requires that  $z \gg \zeta^{\frac{1}{1-\alpha}}$ , which is the same as  $\chi \gg \tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}}$ . The domination of  $A_l$  in the denominator requires:  $z \gg A_l^{-1/\alpha}$ , which is the same as  $\chi \gg \tau_M^{-1} \zeta^{-1} A_l^{-1/\alpha}$ . It also demands that  $\zeta z^{-1} \ll A_l$ , which is:  $\chi \gg \tau_M^{-1} A_l^{-1}$ .

For realistic values of  $A_l$  (say,  $10^3$ ) and  $\alpha$  (say, 0.25), we have:  $A_l^{-1/\alpha} \sim 10^{-12}$ . At high frequencies, anelasticity beats viscosity, so  $\zeta$  is less than unity (or, at least, is not much larger than unity). On these grounds, the requirement  $\chi \gg \tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}}$  is the strongest here. Its fulfilment guarantees that of both  $\chi \gg \tau_M^{-1} \zeta^{-1} A_l^{-1/\alpha}$  and  $\chi \gg \tau_M/A_l$ . Thus we have:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) \approx \frac{3}{2(l-1)} \frac{1}{A_l} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha+1) \zeta^{-\alpha} (\tau_M \chi)^{-\alpha} \quad \text{for} \quad \chi \gg \tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}}. \quad (257a)$$

### E.2.3 The intermediate band

Within the intermediate band, the term  $\zeta z^{-1}$  takes over in the numerator, while  $A_l$  still dominates in the denominator. The domination of  $\zeta z^{-\alpha}$  in the numerator implies that  $z \ll \zeta^{\frac{1}{1-\alpha}}$ , which is equivalent to  $\chi \gg \tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}}$ . The domination of  $A_l$  in the denominator requires  $\chi \gg \tau_M^{-1} \zeta^{-1} A_l^{-1/\alpha}$  and  $\chi \gg \tau_M^{-1} A_l^{-1}$ , as we just saw above.

As we are considering the band where viscosity takes over anelasticity, we may expect that here  $\zeta$  is about or, likely, larger than unity. Taken the large value of  $A_l$ , we see that the condition  $\chi \gg \tau_M^{-1} A_l^{-1}$  is stronger. Its fulfilment guarantees the fulfilment of  $\chi \gg \tau_M^{-1} \zeta^{-1} A_l^{-1/\alpha}$ . This way, we obtain:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) \approx \frac{3}{2(l-1)} \frac{1}{A_l} (\tau_M \chi)^{-1} \quad \text{for} \quad \tau_M^{-1} \zeta^{\frac{\alpha}{1-\alpha}} \gg \chi \gg \tau_M^{-1} A_l^{-1} . \quad (257b)$$

### E.2.4 The low-frequency band

For frequencies lower than  $\tau_M^{-1} A_l^{-1}$  the Andrade model renders the same frequency-dependency as that given (at frequencies below  $\tau_M^{-1}$ ) by the Maxwell model:

$$|\bar{k}_l(\chi)| \sin \epsilon_l(\chi) \approx \frac{3}{2(l-1)} A_l \tau_M \chi \quad \text{for} \quad \tau_M^{-1} A_l^{-1} \gg \chi . \quad (257c)$$

### E.2.5 Interpretation

Formulae (257a) and (257b) render a frequency-dependence mimicking that of  $|\bar{J}(\chi)| \sin \delta(\chi) = -\mathcal{I}m[\bar{J}(\chi)]$  in the high- and low-frequency bands. This can be seen from comparing (257a) and (257b) with (248).

In contrast, (257c) reveals a peculiar feature inherent in the *tidal* lagging, and absent in the lagging in a sample.

For terrestrial bodies, the condition  $\tau_M^{-1} A_l^{-1} \gg \chi$  puts the values of  $\chi$  below  $10^{-10} Hz$ , give or take several orders of magnitude. Hence  $|\bar{k}_l(\chi)| \sin \epsilon_l(\chi)$  follows the linear scaling law (257c) only in an extremely close vicinity of the commensurability where the frequency  $\chi$  vanishes. Nonetheless it is very important that  $|\bar{k}_2(\chi)| \sin \epsilon(\chi)$  first reaches a finite maximum and then decreases continuously and vanishes, as the frequency goes to zero. This confirms that neither the tidal torque nor the tidal force becomes infinite in resonances.

## F The behaviour of $k_l(\chi) \equiv |\bar{k}_l(\chi)|$ in the limit of vanishing tidal frequency $\chi$ , within the Andrade and Maxwell models

From (68), we obtain:

$$|\bar{k}_l(\chi)|^2 = \left( \frac{3}{2(l-1)} \right)^2 \frac{|\bar{J}(\chi)|}{|A_l J + \bar{J}(\chi)|} = \left( \frac{3}{2(l-2)} \right)^2 \frac{\left( \mathcal{R}e[\bar{J}(\chi)] \right)^2 + \left( \mathcal{I}m[\bar{J}(\chi)] \right)^2}{\left( \mathcal{R}e[\bar{J}(\chi)] + A_l J \right)^2 + \left( \mathcal{I}m[\bar{J}(\chi)] \right)^2} . \quad (258)$$

Bringing in expressions for the imaginary and real parts of the compliance, and introducing notations

$$E \equiv \eta^{-1} , \quad B \equiv \beta \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(1+\alpha) , \quad D \equiv \beta \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(1+\alpha) , \quad (259)$$

we can write:

$$|k_l(\chi)|^2 = \left( \frac{3}{2(l-1)} \right)^2 \left[ 1 - \frac{2 A_l J D}{E^2} \chi^{2-\alpha} - \frac{2 A_l J^2 + A_l^2 J^2}{E^2} \chi^2 + O(\chi^{3-2\alpha}) \right] \quad (260)$$

and

$$|k_l(\chi)|^{-2} = \left( \frac{2(l-1)}{3} \right)^2 \left[ 1 + \frac{2 A_l J D}{E^2} \chi^{2-\alpha} + \frac{2 A_l J^2 + A_l^2 J^2}{E^2} \chi^2 + O(\chi^{3-2\alpha}) \right] \quad (261)$$

whence

$$|k_l(\chi)| = \frac{3}{2(l-1)} \left[ 1 - A_l J \frac{D}{E^2} \chi^{2-\alpha} - A_l J \frac{J + A_l J/2}{E^2} \chi^2 + O(\chi^{3-2\alpha}) \right] \quad (262)$$

and

$$|k_l(\chi)|^{-1} = \frac{2(l-1)}{3} \left[ 1 + A_l J \frac{D}{E^2} \chi^{2-\alpha} + A_l J \frac{J + A_l J/2}{E^2} \chi^2 + O(\chi^{3-2\alpha}) \right] \quad (263)$$

the expansions being valid for  $\chi J/E \ll 1 + A_l$ , i.e., for  $\chi \tau_M \ll 1 + A_l$ .

Rewriting (68) as

$$\bar{k}_l(\chi) = \frac{3}{2(l-1)} \frac{\left( \mathcal{R}e \left[ \bar{J}(\chi) \right] + i \mathcal{I}m \left[ \bar{J}(\chi) \right] \right) \left( \mathcal{R}e \left[ \bar{J}(\chi) \right] + A_l J - i \mathcal{I}m \left[ \bar{J}(\chi) \right] \right)}{\left( \mathcal{R}e \left[ \bar{J}(\chi) \right] + A_l J \right)^2 + \left( \mathcal{I}m \left[ \bar{J}(\chi) \right] \right)^2} \quad (264)$$

we extract its real part:

$$\begin{aligned} \mathcal{R}e \left[ \bar{k}_l(\chi) \right] &= \frac{3}{2(l-1)} \frac{\left( \mathcal{R}e \left[ \bar{J}(\chi) \right] \right)^2 + \left( \mathcal{I}m \left[ \bar{J}(\chi) \right] \right)^2 + A_l J \mathcal{R}e \left[ \bar{J}(\chi) \right]}{\left( \mathcal{R}e \left[ \bar{J}(\chi) \right] + A_l J \right)^2 + \left( \mathcal{I}m \left[ \bar{J}(\chi) \right] \right)^2} \\ &= \frac{3}{2(l-1)} \left[ 1 - A_l J \frac{\mathcal{R}e \left[ \bar{J}(\chi) \right] + A_l J}{\left( \mathcal{R}e \left[ \bar{J}(\chi) \right] + A_l J \right)^2 + \left( \mathcal{I}m \left[ \bar{J}(\chi) \right] \right)^2} \right] \quad (265) \end{aligned}$$

Insertion of the expressions for  $\mathcal{R}e \left[ \bar{J}(\chi) \right]$  and  $\mathcal{I}m \left[ \bar{J}(\chi) \right]$  into the latter formula entails:

$$\mathcal{R}e \left[ \bar{k}_l(\chi) \right] = \frac{3}{2(l-1)} \left[ 1 - A_l J \frac{D}{E^2} \chi^{2-\alpha} - A_l J \frac{J + A_l J}{E^2} \chi^2 + O(\chi^{3-2\alpha}) \right] \quad (266)$$

Expressions (263) and (266) enable us to write down the cosine of the shape lag:

$$\cos \epsilon_l = \frac{\mathcal{R}e[k_l(\chi)]}{|k_l(\chi)|} = 1 - \frac{1}{2} \left( \frac{A_l J}{E} \right)^2 \chi^2 + O(\chi^{3-2\alpha}) = 1 - \frac{1}{2} A_l^2 (\tau_M \chi)^2 + O(\chi^{3-2\alpha}) \quad (267)$$

Comparing this expression with (262), we see that, for the Andrade ( $\alpha \neq 0$ ) model, the evolution of  $k_l(\chi) \equiv |\bar{k}_2(\chi)|$  in the limit of small  $\chi$ , unfortunately, cannot be approximated with a convenient expression  $k_l(\chi) \approx k_l(0) \cos \epsilon(\chi)$ , which is valid for simpler models (like the one of Kelvin-Voigt or SAS).

However, for the Maxwell model ( $\beta = 0$ ) expression (262) becomes:

$$|k_l(\chi)| = \frac{3}{2(l-1)} \left[ 1 - \frac{A_l J}{E^2} (J + A_l J/2) \chi^2 + O(\chi^3) \right] = \frac{3}{2(l-1)} [1 - A_l (1 + A_l/2) (\tau_M \chi)^2 + O(\chi^3)] \quad (268)$$

which can be written as

$$|k_l(\chi)| \approx \frac{3}{2(l-1)} \left[ 1 - \frac{1}{2} \left( \frac{A_l J}{E} \right)^2 \chi^2 + O(\chi^3) \right] = \frac{3}{2(l-1)} \left[ 1 - \frac{1}{2} A_l^2 (\tau_M \chi)^2 + O(\chi^3) \right] , \quad (269)$$

insofar as  $A_l \gg 1$ . Comparing this with (267), we see that, for small terrestrial moons and planets (but not for superearths whose  $A_l$  is small), the following convenient approximation is valid, provided the Maxwell model is employed:

$$k_l(\chi) \approx k_l(0) \cos \epsilon(\chi) \quad , \quad \text{for} \quad \chi \tau_M \ll 1 + A_l \quad \text{where} \quad A_l \gg 1 \quad . \quad (270)$$

## G The eccentricity functions

In our development, we take into account the expansions for  $G_{lpq}^2(e)$  over the powers of eccentricity, keeping the terms up to  $e^6$ , inclusively. The table of the eccentricity functions presented in the book by Kaula (1966) is not sufficient for our purposes, because some of the  $G_{lpq}(e)$  functions in that table are given with lower precision. For example, the  $e^6$  term is missing in the approximation for  $G_{200}(e)$ . Besides, that table omits several functions which are of order  $e^3$ . So here we provide a more comprehensive table. The table is based on the information borrowed from Cayley (1861) who tabulated various expansions employed in astronomy. Among those, were series

$$\left( \frac{r}{a} \right)^{-(l+1)} \begin{bmatrix} [\cos]^i & \cos \\ [\sin]^i & \sin \end{bmatrix} j\nu = \sum_{i=-\infty}^{\infty} \begin{bmatrix} \cos \\ \sin \end{bmatrix} i\mathcal{M} \quad , \quad (271)$$

$\nu$  and  $\mathcal{M}$  signifying the true and mean anomalies, while  $[\cos]^i$  and  $[\sin]^i$  denoting the coefficients tabulated by Cayley. These coefficients are polynomials in the eccentricity. Cayley's integer indices  $i, j$  are connected with Kaula's integers  $l, p, q$  via

$$l - 2p = j \quad , \quad l - 2p + q = i \quad . \quad (272)$$

With the latter equalities kept in mind, the eccentricity functions, for  $i \geq 0$ , are related to Cayley's coefficients by

$$G_{lpq}(e) = [\cos]^i + [\sin]^i \quad , \quad \text{for} \quad i \geq 0 \quad . \quad (273)$$

To obtain the eccentricity functions for  $i < 0$ , one has to keep in mind that  $[\cos]^{-i} = [\cos]^i$ , while  $[\sin]^i = -[\sin]^{-i}$ . It is then possible to demonstrate that

$$G_{lpq}(e) = [\cos]^i - [\sin]^i \quad , \quad \text{for} \quad i < 0 \quad . \quad (274)$$

Then the following expressions, for  $l = 2$ , ensue from Cayley's tables:

$$G_{20 \ -11}(e) = G_{20 \ -10}(e) = G_{20 \ -9}(e) = G_{20 \ -8}(e) = 0 \quad , \quad (275a)$$

$$G_{20 \ -7}(e) = \frac{15625}{129024} e^7 \quad , \quad (275b)$$

$$G_{20 \ -6}(e) = \frac{4}{45} e^6 \quad , \quad (275c)$$

$$G_{20 \ -5}(e) = \frac{81}{1280} e^5 + \frac{81}{2048} e^7 \quad , \quad (275d)$$

$$G_{20 \ -4}(e) = \frac{1}{24} e^4 + \frac{7}{240} e^6 \quad , \quad (275e)$$

$$G_{20 \ -3}(e) = \frac{1}{48} e^3 + \frac{11}{768} e^5 + \frac{313}{30720} e^7 \quad , \quad (275f)$$

$$G_{20 \ -2}(e) = 0 \quad , \quad (275g)$$

$$G_{20 \ -1}(e) = -\frac{1}{2} e + \frac{1}{16} e^3 - \frac{5}{384} e^5 - \frac{143}{18432} e^7 \quad , \quad (275h)$$

$$G_{200}(e) = 1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 - \frac{35}{288} e^6 \quad , \quad (275i)$$

$$G_{201}(e) = \frac{7}{2} e - \frac{123}{16} e^3 + \frac{489}{128} e^5 - \frac{1763}{2048} e^7 \quad (275j)$$

$$G_{202}(e) = \frac{17}{2} e^2 - \frac{115}{6} e^4 + \frac{601}{48} e^6 \quad , \quad (275k)$$

$$G_{203}(e) = \frac{845}{48} e^3 - \frac{32525}{768} e^5 + \frac{208225}{6144} e^7 \quad , \quad (275l)$$

$$G_{204}(e) = \frac{533}{16} e^4 - \frac{13827}{160} e^6 \quad , \quad (275m)$$

$$G_{205}(e) = \frac{228347}{3840} e^5 - \frac{3071075}{18432} e^7 \quad , \quad (275n)$$

$$G_{206}(e) = \frac{73369}{720} e^6 \quad , \quad (275o)$$

$$G_{207}(e) = \frac{12144273}{71680} e^7 \quad , \quad (275p)$$

the other values of  $q$  generating polynomials  $G_{20q}(e)$  whose leading terms are of order  $e^8$  and higher.

Since in our study we intend to employ the squares of these functions, with terms up to  $e^6$  only, then we may completely omit the eccentricity functions with  $|q| \geq 4$ . In our approximation,



the squares of the eccentricity functions will look:

$$G_{20-3}^2(e) = \frac{1}{2304} e^6 + O(e^8) , \quad (276a)$$

$$G_{20-2}^2(e) = 0 , \quad (276b)$$

$$G_{20-1}^2(e) = \frac{1}{4} e^2 - \frac{1}{16} e^4 + \frac{13}{768} e^6 + O(e^8) , \quad (276c)$$

$$G_{200}^2(e) = 1 - 5 e^2 + \frac{63}{8} e^4 - \frac{155}{36} e^6 + O(e^8) , \quad (276d)$$

$$G_{201}^2(e) = \frac{49}{4} e^2 - \frac{861}{16} e^4 + \frac{21975}{256} e^6 + O(e^8) , \quad (276e)$$

$$G_{202}^2(e) = \frac{289}{4} e^4 - \frac{1955}{6} e^6 + O(e^8) , \quad (276f)$$

$$G_{203}^2(e) = \frac{714025}{2304} e^6 + O(e^8) , \quad (276g)$$

the squares of the others being of the order of  $e^8$  or higher.

Be mindful that, for  $l = 2$  we considered only the functions with  $p = 0$ . This is dictated by the fact that the inclination functions  $F_{lmp} = F_{22p}$  are of order  $i$  (and, accordingly, their squares and cross-products are of order  $i^2$ ) for all the values of  $p$  except zero.

For  $l = 3$ , the situation changes. The inclination functions  $F_{lmp} = F_{310}, F_{312}, F_{313}, F_{320}, F_{321}, F_{322}, F_{323}, F_{331}, F_{332}, F_{333}$  are of the order  $O(i)$  or higher. The terms containing the squares or cross-products of these functions may thus be omitted. (Specifically, by neglecting the cross-terms we get rid of the mixed-period part of the  $l = 3$  component.) What is left is the terms with  $lmp = 311$  and  $lmp = 330$ . These terms contain the squares of functions

$$F_{311}(i) = -\frac{3}{2} + O(i^2) \quad \text{and} \quad F_{330}(i) = 15 + O(i^2) , \quad (277)$$

accordingly. From here, we see that, for  $l = 3$ , we shall need to employ the eccentricity functions  $G_{lpq}(e) = G_{30q}(e)$  and  $G_{lpq}(e) = G_{31q}(e)$ .

The following expressions, for  $l = 3$  and  $p = 0$ , ensue from Cayley's tables:

$$G_{30-11}(e) = G_{30-10}(e) = G_{30-9}(e) = G_{30-8}(e) = 0 , \quad (278a)$$

$$G_{30-7}(e) = \frac{8}{315} e^7 , \quad (278b)$$

$$G_{30-6}(e) = \frac{81}{5120} e^6 , \quad (278c)$$

$$G_{30-5}(e) = \frac{1}{120} e^5 + \frac{13}{1440} e^7 , \quad (278d)$$

$$G_{30-4}(e) = \frac{1}{384} e^4 + \frac{1}{384} e^6 , \quad (278e)$$

$$G_{30-3}(e) = 0 \quad , \quad (278f)$$

$$G_{30-2}(e) = \frac{1}{8} e^2 + \frac{1}{48} e^4 + \frac{55}{3072} e^6 \quad , \quad (278g)$$

$$G_{30-1}(e) = -e + \frac{5}{4} e^3 - \frac{7}{48} e^5 + \frac{23}{288} e^7 \quad , \quad (278h)$$

$$G_{300}(e) = 1 - 6 e^2 + \frac{423}{64} e^4 - \frac{125}{64} e^6 \quad , \quad (278i)$$

$$G_{301}(e) = 5 e - 22 e^3 + \frac{607}{24} e^5 - \frac{98}{9} e^7 \quad (278j)$$

$$G_{302}(e) = \frac{127}{8} e^2 - \frac{3065}{48} e^4 + \frac{243805}{3072} e^6 \quad , \quad (278k)$$

$$G_{303}(e) = \frac{163}{4} e^3 - \frac{2577}{16} e^5 + \frac{1089}{5} e^7 \quad , \quad (278l)$$

$$G_{304}(e) = \frac{35413}{384} e^4 - \frac{709471}{1920} e^6 \quad , \quad (278m)$$

$$G_{305}(e) = \frac{23029}{120} e^5 - \frac{35614}{45} e^7 \quad , \quad (278n)$$

$$G_{306}(e) = \frac{385095}{1024} e^6 \quad , \quad (278o)$$

$$G_{307}(e) = \frac{44377}{63} e^7 \quad , \quad (278p)$$

the other values of  $q$  generating polynomials  $G_{30q}(e)$  and  $G_{31q}(e)$ , whose leading terms are of order  $e^8$  and higher.

The squares of some these functions, will read, up to  $e^6$  terms inclusively, as:

$$G_{30-3}^2(e) = 0 \quad , \quad (279a)$$

$$G_{30-2}^2(e) = \frac{1}{64} e^4 + \frac{1}{192} e^6 + O(e^8) \quad , \quad (279b)$$

$$G_{30-1}^2(e) = e^2 - \frac{5}{2} e^4 + \frac{89}{48} e^6 + O(e^8) \quad , \quad (279c)$$

$$G_{300}^2(e) = 1 - 12 e^2 + \frac{1575}{32} e^4 - \frac{2663}{32} e^6 + O(e^8) \quad , \quad (279d)$$

$$G_{301}^2(e) = 25 e^2 - 220 e^4 + \frac{8843}{12} e^6 + O(e^8) \quad , \quad (279e)$$

$$G_{302}^2(e) = \frac{16129}{64} e^4 - \frac{389255}{192} e^6 + O(e^8) \quad , \quad (279f)$$

$$G_{303}^2(e) = \frac{26569}{16} e^6 + O(e^8) \quad , \quad (279g)$$

the squares of the others being of the order of  $e^8$  or higher.

Finally, we write down the expressions for the eccentricity functions with  $l = 3$  and  $p = 1$ :

$$G_{31-9}(e) = G_{31-8}(e) = 0 \quad , \quad (280a)$$

$$G_{31-7}(e) = \frac{16337}{2240} e^7 \quad , \quad (280b)$$

$$G_{31-6}(e) = \frac{48203}{9216} e^6 \quad , \quad (280c)$$

$$G_{31-5}(e) = \frac{899}{240} e^5 + \frac{2441}{480} e^7 \quad , \quad (280d)$$

$$G_{31-4}(e) = \frac{343}{128} e^4 + \frac{2819}{640} e^6 \quad , \quad (280e)$$

$$G_{31-3}(e) = \frac{23}{12} e^3 + \frac{89}{24} e^5 + \frac{5663}{960} e^7 \quad , \quad (280f)$$

$$G_{31-2}(e) = \frac{11}{8} e^2 + \frac{49}{16} e^4 + \frac{15665}{3072} e^6 \quad , \quad (280g)$$

$$G_{31-1}(e) = e + \frac{5}{2} e^3 + \frac{35}{8} e^5 + \frac{105}{16} e^7 \quad , \quad (280h)$$

$$G_{310}(e) = 1 + 2 e^2 + \frac{239}{64} e^4 + \frac{3323}{576} e^6 \quad , \quad (280i)$$

$$G_{311}(e) = 3 e + \frac{11}{4} e^3 + \frac{245}{48} e^5 + \frac{463}{64} e^7 \quad (280j)$$

$$G_{312}(e) = \frac{53}{8} e^2 + \frac{39}{16} e^4 + \frac{7041}{1024} e^6 \quad , \quad (280k)$$

$$G_{313}(e) = \frac{77}{6} e^3 - \frac{25}{48} e^5 + \frac{4751}{480} e^7 \quad , \quad (280l)$$

$$G_{314}(e) = \frac{2955}{128} e^4 - \frac{3463}{384} e^6 \quad , \quad (280m)$$

$$G_{315}(e) = \frac{3167}{80} e^5 - \frac{8999}{320} e^7 \quad , \quad (280n)$$

$$G_{316}(e) = \frac{3024637}{46080} e^6 \quad , \quad (280o)$$

$$G_{317}(e) = \frac{178331}{1680} e^7 \quad , \quad (280p)$$

and the squares:

$$G_{31-3}^2(e) = \frac{529}{144} e^6 + O(e^8) \quad , \quad (281a)$$

$$G_{31-2}^2(e) = \frac{121}{64} e^4 + \frac{539}{64} e^6 + O(e^8) \quad , \quad (281b)$$

$$G_{31-1}^2(e) = e^2 + 5 e^4 + 15 e^6 + O(e^8) \quad , \quad (281c)$$

$$G_{310}^2(e) = 1 + 4 e^2 + \frac{367}{32} e^4 + \frac{7625}{288} e^6 + O(e^8) \quad , \quad (281d)$$

$$G_{311}^2(e) = 9 e^2 + \frac{33}{2} e^4 + \frac{611}{16} e^6 + O(e^8) \quad , \quad (281e)$$

$$G_{312}^2(e) = \frac{2809}{64} e^4 + \frac{2067}{64} e^6 + O(e^8) \quad , \quad (281f)$$

$$G_{313}^2(e) = \frac{5929}{36} e^6 + O(e^8) \quad , \quad (281g)$$

the squares of the other functions from this set being of the order  $e^8$  or higher.

## H The $l = 2$ and $l = 3$ terms of the secular part of the torque

### H.1 The $l = 2$ terms of the secular torque

Extracting the  $l = 2$  input from (113), we recall that only the  $(lmpq) = (220q)$  terms matter. Out of these terms, we need only the ones up to  $e^6$ . These are the terms with  $|q| \leq 3$ . They are given by formulae (276) from Appendix G. Employing those formulae, we arrive at

$$\overline{\mathcal{T}}_{l=2} = \overline{\mathcal{T}}_{(lmp)=(220)} + O(i^2 \epsilon) = \frac{3}{2} G M_{sec}^2 R^5 a^{-6} \sum_{q=-3}^3 G_{20q}^2(e) k_2 \sin \epsilon_{220q} + O(e^8 \epsilon) + O(i^2 \epsilon) \quad (282a)$$

$$\begin{aligned} &= \frac{3}{2} G M_{sec}^2 R^5 a^{-6} \left[ \frac{1}{2304} e^6 k_2 \sin \epsilon_{220-3} + \left( \frac{1}{4} e^2 - \frac{1}{16} e^4 + \frac{13}{768} e^6 \right) k_2 \sin \epsilon_{220-1} \right. \\ &+ \left( 1 - 5 e^2 + \frac{63}{8} e^4 - \frac{155}{36} e^6 \right) k_2 \sin \epsilon_{2200} + \left( \frac{49}{4} e^2 - \frac{861}{16} e^4 + \frac{21975}{256} e^6 \right) k_2 \sin \epsilon_{2201} \\ &+ \left. \left( \frac{289}{4} e^4 - \frac{1955}{6} e^6 \right) k_2 \sin \epsilon_{2202} + \frac{714025}{2304} e^6 k_2 \sin \epsilon_{2203} \right] + O(e^8 \epsilon) + O(i^2 \epsilon) \quad , \quad (282b) \end{aligned}$$

where the absolute error  $O(e^8 \epsilon)$  has emerged because of our neglect of terms with  $|q| \geq 4$ , while the absolute error  $O(i^2 \epsilon)$  came into being after the truncation of terms with  $p \geq 1$ .

Recalling expression (109b), we can rewrite (282b) in a form indicating explicitly at which resonance each term changes its sign. To this end, each  $k_l \sin \epsilon_{lmpq}$  will be rewritten as:

$k_l \sin |\epsilon_{lmpq}| \operatorname{sgn} \left[ (l - 2p + q) n - m \dot{\theta} \right]$  . This will render:

$$\begin{aligned}
\overline{\mathcal{T}}_{l=2} = & \frac{3}{2} G M_{sec}^2 R^5 a^{-6} \left[ \frac{1}{2304} e^6 k_2 \sin |\epsilon_{220-3}| \operatorname{sgn} (-n - 2\dot{\theta}) \right. \\
& + \left( \frac{1}{4} e^2 - \frac{1}{16} e^4 + \frac{13}{768} e^6 \right) k_2 \sin |\epsilon_{220-1}| \operatorname{sgn} (n - 2\dot{\theta}) \\
& + \left( 1 - 5e^2 + \frac{63}{8} e^4 - \frac{155}{36} e^6 \right) k_2 \sin |\epsilon_{2200}| \operatorname{sgn} (n - \dot{\theta}) \\
& + \left( \frac{49}{4} e^2 - \frac{861}{16} e^4 + \frac{21975}{256} e^6 \right) k_2 \sin |\epsilon_{2201}| \operatorname{sgn} (3n - 2\dot{\theta}) \\
& + \left( \frac{289}{4} e^4 - \frac{1955}{6} e^6 \right) k_2 \sin |\epsilon_{2202}| \operatorname{sgn} (2n - \dot{\theta}) \\
& \left. + \frac{714025}{2304} e^6 k_2 \sin |\epsilon_{2203}| \operatorname{sgn} (5n - 3\dot{\theta}) \right] + O(e^8 \epsilon) + O(i^2 \epsilon) \quad , \quad (283)
\end{aligned}$$

## H.2 The $l = 3$ , $m = 1$ terms of the secular torque

Getting the  $l = 3$ ,  $m = 1$  input from (113) and leaving in it only the terms up to  $e^6$ , we obtain, with aid of formulae (276) from Appendix G, the following expression:

$$\overline{\mathcal{T}}_{(lmp)=(311)} = \frac{3}{8} G M_{sec}^2 R^7 a^{-8} \sum_{q=-3}^3 G_{31q}^2(e) k_3 \sin \epsilon_{311q} + O(e^8 \epsilon) \quad (284a)$$

$$\begin{aligned}
= & \frac{3}{8} G M_{sec}^2 R^7 a^{-8} \left[ \frac{529}{144} e^6 k_3 \sin \epsilon_{311-3} + \left( \frac{121}{64} e^4 + \frac{539}{64} e^6 \right) k_3 \sin \epsilon_{311-2} \right. \\
& + (e^2 + 5e^4 + 15e^6) k_3 \sin \epsilon_{311-1} + \left( 1 + 4e^2 + \frac{367}{32} e^4 + \frac{7625}{288} e^6 \right) k_3 \sin \epsilon_{3110} \\
& + \left( 9e^2 + \frac{33}{2} e^4 + \frac{611}{16} e^6 \right) k_3 \sin \epsilon_{3111} + \left( \frac{2809}{64} e^4 + \frac{2067}{64} e^6 \right) k_3 \sin \epsilon_{3112} \\
& \left. + \frac{5929}{36} e^6 k_3 \sin \epsilon_{3113} \right] + O(e^8 \epsilon) \quad . \quad (284b)
\end{aligned}$$

With the signs depicted explicitly, this will look:

$$\begin{aligned}
\overline{\mathcal{T}}_{(lmp)=(311)} &= \frac{3}{8} G M_{sec}^2 R^7 a^{-8} \left[ \frac{529}{144} e^6 k_3 \sin |\epsilon_{311-3}| \operatorname{sgn}(-2n - \dot{\theta}) \right. \\
&+ \left( \frac{121}{64} e^4 + \frac{539}{64} e^6 \right) k_3 \sin |\epsilon_{311-2}| \operatorname{sgn}(-n - \dot{\theta}) \\
&+ (e^2 + 5e^4 + 15e^6) k_3 \sin |\epsilon_{311-1}| \operatorname{sgn}(-\dot{\theta}) \\
&+ \left( 1 + 4e^2 + \frac{367}{32} e^4 + \frac{7625}{288} e^6 \right) k_3 \sin |\epsilon_{3110}| \operatorname{sgn}(n - \dot{\theta}) \\
&+ \left( 9e^2 + \frac{33}{2} e^4 + \frac{611}{16} e^6 \right) k_3 \sin |\epsilon_{3111}| \operatorname{sgn}(2n - \dot{\theta}) \\
&+ \left( \frac{2809}{64} e^4 + \frac{2067}{64} e^6 \right) k_3 \sin |\epsilon_{3112}| \operatorname{sgn}(3n - \dot{\theta}) \\
&\left. + \frac{5929}{36} e^6 k_3 \sin |\epsilon_{3113}| \operatorname{sgn}(4n - \dot{\theta}) \right] + O(e^8 \epsilon) . \tag{285}
\end{aligned}$$

### H.3 The $l = 3$ , $m = 3$ terms of the secular torque

The second relevant group of terms with  $l = 3$  will read:

$$\overline{\mathcal{T}}_{(lmp)=(330)} = \frac{15}{8} G M_{sec}^2 R^7 a^{-8} \sum_{q=-3}^3 G_{30q}^2(e) k_3 \sin \epsilon_{330q} + O(e^8 \epsilon) \tag{286a}$$

$$\begin{aligned}
&= \frac{15}{8} G M_{sec}^2 R^7 a^{-8} \left[ \left( \frac{1}{64} e^4 + \frac{1}{192} e^6 \right) k_3 \sin \epsilon_{330-2} \right. \\
&+ \left( e^2 - \frac{5}{2} e^4 + \frac{89}{48} e^6 \right) k_3 \sin \epsilon_{330-1} + \left( 1 - 12e^2 + \frac{1575}{32} e^4 - \frac{2663}{32} e^6 \right) k_3 \sin \epsilon_{3300} \\
&+ \left( 25e^2 - 220e^4 + \frac{8843}{12} e^6 \right) k_3 \sin \epsilon_{3301} + \left( \frac{16129}{64} e^4 - \frac{389255}{192} e^6 \right) k_3 \sin \epsilon_{3302} \\
&\left. + \frac{26569}{16} e^6 k_3 \sin \epsilon_{3303} \right] + O(e^8 \epsilon) \tag{286b}
\end{aligned}$$

or, with the signs shown explicitly:

$$\begin{aligned}
\bar{\mathcal{T}}_{(lmp)=(330)} &= \frac{15}{8} G M_{sec}^2 R^7 a^{-8} \left[ \left( \frac{1}{64} e^4 + \frac{1}{192} e^6 \right) k_3 \sin |\epsilon_{330-2}| \operatorname{sgn} \left( -n - \dot{\theta} \right) \right. \\
&+ \left( e^2 - \frac{5}{2} e^4 + \frac{89}{48} e^6 \right) k_3 \sin |\epsilon_{330-1}| \operatorname{sgn} \left( -\dot{\theta} \right) \\
&+ \left( 1 - 12 e^2 + \frac{1575}{32} e^4 - \frac{2663}{32} e^6 \right) k_3 \sin |\epsilon_{3300}| \operatorname{sgn} \left( n - \dot{\theta} \right) \\
&+ \left( 25 e^2 - 220 e^4 + \frac{8843}{12} e^6 \right) k_3 \sin |\epsilon_{3301}| \operatorname{sgn} \left( 2n - \dot{\theta} \right) \\
&+ \left( \frac{16129}{64} e^4 - \frac{389255}{192} e^6 \right) k_3 \sin |\epsilon_{3302}| \operatorname{sgn} \left( 3n - \dot{\theta} \right) \\
&\left. + \frac{26569}{16} e^6 k_3 \sin |\epsilon_{3303}| \operatorname{sgn} \left( 4n - \dot{\theta} \right) \right] + O(e^8 \epsilon) \quad . \quad (287)
\end{aligned}$$

## I The $l = 2$ and $l = 3$ terms of the short-period part of the torque

The short-period part of the torque may be approximated with terms of degrees 2 and 3:

$$\begin{aligned}
\tilde{\mathcal{T}} &= \tilde{\mathcal{T}}_{l=2} + \tilde{\mathcal{T}}_{l=3} + O(\epsilon(R/a)^9) \\
&= \tilde{\mathcal{T}}_{(lmp)=(220)} + \left[ \tilde{\mathcal{T}}_{(lmp)=(311)} + \tilde{\mathcal{T}}_{(lmp)=(330)} \right] + O(\epsilon i^2) + O(\epsilon(R/a)^9) \quad , \quad (288)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{T}}_{(lmp)=(220)} &= 3 G M_{sec}^2 R^5 a^{-6} \sum_{q=-3}^3 \sum_{\substack{j=-3 \\ j < q}}^3 G_{20q}(e) G_{20j}(e) \left\{ \cos [\mathcal{M}(q-j)] k_2 \sin \epsilon_{220q} \right. \\
&\quad \left. - \sin [\mathcal{M}(q-j)] k_2 \cos \epsilon_{220q} \right\} + O(i^2 \epsilon) + O(e^7 \epsilon) \quad , \quad (289a)
\end{aligned}$$

$$= -3 G M_{sec}^2 R^5 a^{-6} \sum_{q=-3}^3 \sum_{\substack{j=-3 \\ j < q}}^3 G_{20q}(e) G_{20j}(e) k_2 \sin [\mathcal{M}(q-j)] + O(i^2 \epsilon) + O(e \epsilon) \quad , \quad (289b)$$

$$\begin{aligned} \tilde{\mathcal{T}}_{(lmp)=(311)} = & \frac{3}{4} G M_{sec}^2 R^7 a^{-8} \sum_{q=-3}^3 \sum_{\substack{j=-3 \\ j < q}}^3 G_{31q}(e) G_{31j}(e) \left\{ \cos[\mathcal{M}(q-j)] k_3 \sin \epsilon_{311q} \right. \\ & \left. - \sin[\mathcal{M}(q-j)] k_2 \cos \epsilon_{311q} \right\} + O(i^2 \epsilon) + O(e^7 \epsilon) , \end{aligned} \quad (290a)$$

$$= -\frac{3}{4} G M_{sec}^2 R^5 a^{-6} \sum_{q=-3}^3 \sum_{\substack{j=-3 \\ j < q}}^3 G_{31q}(e) G_{31j}(e) k_2 \sin[\mathcal{M}(q-j)] + O(i^2 \epsilon) + O(e \epsilon) , \quad (290b)$$

$$\begin{aligned} \tilde{\mathcal{T}}_{(lmp)=(330)} = & \frac{15}{4} G M_{sec}^2 R^7 a^{-8} \sum_{q=-3}^3 \sum_{\substack{j=-3 \\ j < q}}^3 G_{30q}(e) G_{30j}(e) \cos[\mathcal{M}(q-j)] k_3 \sin \epsilon_{330q} \\ & - \sin[\mathcal{M}(q-j)] k_2 \cos \epsilon_{330q} \left\} + O(i^2 \epsilon) + O(e^7 \epsilon) , \end{aligned} \quad (291a)$$

$$= -\frac{15}{4} G M_{sec}^2 R^5 a^{-6} \sum_{q=-3}^3 \sum_{\substack{j=-3 \\ j < q}}^3 G_{30q}(e) G_{30j}(e) k_2 \sin[\mathcal{M}(q-j)] + O(i^2 \epsilon) + O(e \epsilon) , \quad (291b)$$

the expressions for the eccentricity functions being provided in Appendix G. The overall numerical factors in (289 - 291) are twice the numerical factors in (114), because in (289 - 291) we have  $j < q$  and not  $j \neq q$ . The right-hand sides of (289 - 291) contain  $O(e \epsilon)$  instead of  $O(e^7 \epsilon)$ , because at the final step we approximated  $\cos[\mathcal{M}(q-j)] k_l \sin \epsilon_{lmpq} - \sin[\mathcal{M}(q-j)] k_l \cos \epsilon_{lmpq}$  simply with  $-\sin[\mathcal{M}(q-j)] k_l$ . Doing so, we replaced the cosine with unity, because the entire Darwin-Kaula formalism is a linear approximation in the lags. We also neglected  $k_l \sin \epsilon_{lmpq}$  and kept only the leading term with  $k_l$ . This neglect would be illegitimate in the secular part of the torque, but is probably acceptable in the purely short-period part, because the latter part has a zero average and therefore should be regarded as a small correction even in its leading order. The latter circumstance also will justify approximation of  $k_l = k_l(\chi)$  with  $k_l(0)$  in (289 - 291).

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